Sound Probabilistic Reasoning under Contradiction, Inconsistency and Incompleteness

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Abstract

I intend to show how probability theory can be regarded as logic-dependent, viewing probability as a branch of logic in a generalized way. A kind of meta-axiomatics permits us to define probability measures that are either classical, paraconsistent, intuitionistic, or simultaneously intuitionistic and paraconsistent, just by parameterizing on consequence relations. In particular, I intend to discuss theories of probability built upon the paraconsistent Logic of Formal Inconsistency **Ci**, and upon the paraconsistent and paracomplete Logic of Evidence and Truth **LETj**. I argue that **Ci** very naturally encodes an extension of the notion of probability able to express probabilistic reasoning under an excess of information (contradictions), while **LETj** encodes an extension of the notion of probability able to express probabilistic reasoning under lack of information (incompleteness), and is thus naturally connected to the notion of probability of evidence. I also discuss how interesting non-standard Bayesian updating can be defined in both cases. This is a joint project with J. Bueno-Soler and A. Rodrigues. and most results already appear in [1] and in [5].

1 Consistency versus non-contradictoriness

Paraconsistency is the investigation of logic systems endowed with a negation $\neg$ such that not every contradiction of the form p and $\neg p$ entails everything. In other terms, a paraconsistent logic does not suffer from trivialism, in the sense that a contradiction does not necessarily explode, or trivialize the deductive machinery of the system. In strict terms, even an irrelevant contradiction in traditional logic obliges a reasoner that follows such a logic to derive anything from a contradiction $\alpha, \neg\alpha$, by means of the so-called Principle of Explosion:

(PEx) $\alpha, \neg\alpha, \vdash \beta$, for arbitrary β,

while a paraconsistent logician, by using a more cautious way of reasoning, is free of the burden of (PEx), and could pause to investigate the causes for the contradiction, instead of foolishly deriving unwanted consequences from it.

Common sense, however, recognizes that some contradictions may be indeed intolerable, and those would destroy the very act of reasoning (that is, lead to trivialization). This amounts to recognizing that not all contradictions are equivalent. The Logics of Formal Inconsistency (LFIs), a family of paraconsistent logics designed to express the notion of consistency (and inconsistency as well) within the object language by means of a connective $\circ$ (reading $\circ\alpha$ as "α is consistent") realizes such an intuition.

The distinguishing feature of the LFIs is that the principle of explosion is not valid in general: this principle is not abolished, but restricted to consistent sentences. Therefore, a contradictory theory may be non trivial, unless the contradiction refers to something consistent.

Such features of the LFIs are condensed in the following law, which is referred to as the "Principle of Gentle Explosion":

(PGE) $\circ\alpha, \alpha, \neg\alpha \vdash \beta$, for every β , although $\alpha, \neg\alpha \nvdash \beta$, for some β.

Some philosophers already recognize that it is a mistake to suppose that an inconsistency is the same as a contradiction (e.g., [9]). The LFIS fully formalize this intuition, and starting from this perspective it is possible to build a number of logical systems with different assumptions that not only encode classical reasoning, but also (at the price of adding new principles) converge to classical logic. This note starts from a certain logic endowed with adequate principles to deal with our paraconsistent probability measures, without obfuscating the fact, however, that several other logics would give rise to specific (weaker or stronger) notions of paraconsistent probabilistic measures.

2 Ci a Logic of Formal Inconsistency

Consider the following stock of propositional axioms and rules:

DEFINITION 2.1. *Let Σ be a propositional signature. The logic* **Ci** *(over Σ) is defined by the following Hilbert calculus:*

Axiom Schemas

Ax1. $\alpha \to (\beta \to \alpha)$

Ax2. $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

Ax3. $\alpha \to (\beta \to (\alpha \wedge \beta))$

Ax4. $(\alpha \wedge \beta) \to \alpha$

Ax5. $(\alpha \wedge \beta) \to \beta$

Ax6. $\alpha \to (\alpha \vee \beta)$

Ax7. $\beta \to (\alpha \vee \beta)$

Ax8. $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

Ax9. $\alpha \vee (\alpha \to \beta)$

Ax10. $\alpha \vee \neg\alpha$

Ax11. $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

Ax12. $\neg\neg\alpha \to \alpha$

Ax13. $\alpha \to \neg\neg\alpha$

Ax14. $\neg \circ \alpha \to (\alpha \wedge \neg \alpha)$

Inference Rule

1. *Modus Ponens:* $\alpha, \ \alpha \to \beta \, / \, \beta$

As detailed investigated in [4], **Ci** can be extended to the first-order logic **QCi** (over an appropriate extension of Σ) by adding appropriate axioms and rules. It is worth noting that **Ax1**- **Ax9** plus**MP** define a Hilbert calculus for positive propositional classicallogic (see [3]), and therefore all the laws concerning positive logic (as distribution of $\wedge$ over $\vee$, etc.) are valid.

It is instructive to show, as an example, the useful properties of distribution of conjunction over disjunction that holds as good as in classical logic (not a surprise, since positive classical logic is incorporated into our paraconsistent logic). However, as the validity of such laws may rise some doubts, I provide a quick proof of them (where $\alpha \equiv \beta$ means $\alpha \vdash_{\mathbf{Si}} \beta$ and $\beta \vdash_{\mathbf{Ci}} \alpha$):

THEOREM 2.2.

1. $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
2. $\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

Proof. The easy proof from the axioms is left to the reader. □

As proved in [3], the logic **Ci** cannot be semantically characterized by finite matrices, but it can be characterized in terms of valuations over $\{0,1\}$, or *bivaluations*. As also shown in [3], a strong (classical) negation can be defined in **Ci** as $\sim_\beta \alpha = \alpha \to \bot_\beta$, where $\bot_\beta = (\beta \wedge (\neg \beta \wedge \circ \beta))$ is a bottom formula.[1] for any sentence β. In order to simplify matters, a privileged β will be chosen, and the subscript will be omitted in $\bot_\beta$ and $\sim_\beta$ from now on.

THEOREM 2.3 (Properties of strong negation)**.** *The strong negation $\sim$ of* **Ci** *satisfies all the expected properties of classical negation.*

Proof. All proofs appear in [3]. □

So, for intance, all the following hold where $\vdash$ indicates the derivaility in **Ci**: $\vdash \sim\alpha \to (\alpha \to \psi)$ for every α and ψ; $\vdash \alpha \vee \sim\alpha$; $\vdash \alpha \to \sim\sim\alpha$; $\vdash \sim\sim\alpha \to \alpha$, and $\vdash \bot \to \alpha$

Also, several meta-theorems, as the Deduction Metatheorem, can be proved in **Ci**.

Now by defining α to be inconsistent by $\bullet\alpha := \neg \circ \alpha$ axiom (12) and (13) (which permits to add and eliminate double negations) convey the meaning that ‘ α is not inconsistent if and only if it is consistent’. Of course, by the very definition and the same axioms on double negations, it also holds ‘α is not consistent if and only if it is inconsistent’.

Some other relevant results in **Ci** hold as follows.

THEOREM 2.4. *([3])*

1. $\circ\alpha \vdash \neg(\alpha \wedge \neg\alpha)$*, but the converse does not hold*
2. $\circ\alpha \vdash \neg(\neg\alpha \wedge \alpha)$*, but the converse does not hold*

[1] That is: $\bot_\beta \vdash \psi$ for every ψ.

3. $\vdash \circ\circ\alpha \qquad \vdash \circ\bullet\alpha$

4. $\vdash \neg\bullet\circ\alpha \qquad \vdash \neg\bullet\bullet\alpha$

5. $\vdash \circ\alpha \vee \alpha \qquad \vdash \circ\alpha \vee \neg\alpha$

6. $\vdash \circ\alpha \vee \alpha \wedge \neg\alpha \qquad \alpha \vdash \alpha \wedge (\beta \vee \neg\beta)$

7. $\alpha \wedge (\beta \vee \neg\beta) \vdash \alpha$

THEOREM 2.5. *([3]) The following are bottom particles in* **Ci***:*

1. $\alpha \wedge \neg\alpha \wedge \circ\alpha \vdash \beta$*, for any* β

2. $\circ\alpha \wedge \bullet\alpha \vdash \beta$*, for any* β

3. $\circ\alpha \wedge \neg\circ\alpha \vdash \beta$*, for any* β

4. $\bullet\alpha \wedge \neg\bullet\alpha \vdash \beta$*, for any* β

3 Consistency, inconsistency and paraconsistent probability

As hinted in the previous section, the formal notion of consistency treated here does not depend on negation, and the logical machinery of the LFIs show that consistency may be conceived as a primitive concept, whose meaning can be thought of as "conclusively established as true (or false)", by extra-logical means, depending on the subject matter. Consistency, in this sense, is a notion independent of model theoretical and proof-theoretical means. and is more close to the idea of regularity, or something contrary to change (an elaboration of this view is offered in [2].

The intention here is to introduce a new way to define theories of probability based on some non-classical logics. A previous approach has been developed in [6], who discusses variations of paraconsistent Bayesianism based on a four-valued paraconsistent logic. A still earlier attempt has been briefly sketched in [7]. A completely different view is taken in [8], where probabilistic semantics is given for a couple of many-valued paraconsistent logics.

Probability functions are usually defined for a σ-algebra of subsets of a given universe set Ω, but it is also natural to define probability functions directly for sentences in the object language. They are referred to, respectively, as *probability on sets* versus *probability on sentences*.

Although these two approaches are equivalent in classical logic, due to the representation theorems of propositional logic and other properties of Boolean algebras, this is not so for probability based on other logics, since the algebraic kinship may be lost for non-classical logics, or be much less immediate. Also, in algebraic terms probability functions in set-theoretical settings are required to satisfy countable additivity, but since propositional language is compact, for probability on sentences it suffices to require finite additivity.

DEFINITION 3.1. *A probability function for the language* $\mathscr{L}$ *of a logic* **L***, or a* **L***-probability function, is a function* $P : \mathscr{L} \mapsto \mathbb{R}$ *satisfying the following conditions, where* $\vdash_L$ *stands for the syntactic derivability relation of* **L** *:*

1. *Non-negativity:* $0 \leq P(\varphi) \leq 1$ *for all* $\varphi \in \mathscr{L}$

2. *Tautologicity: If* $\vdash \varphi$*, then* $P(\varphi) = 1$

3. *Anti-Tautologicity: If* $\varphi \vdash_L$*, then* $P(\varphi) = 0$

4. *Comparison: If* $\psi \vdash \varphi$*, then* $P(\psi) \leq P(\varphi)$

5. *Finite Additivity:* $P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\varphi \wedge \psi)$

The meaning of such (meta) axioms can be clarified by noting that Non-Negativity and Finite Additivity are logic-independent axioms; while Tautologicity and Anti-Tautologicity, as well as Comparison, are logic-dependent axioms. Based on this understanding, one may define:

DEFINITION 3.2. *Classical, Intuitionistic, paraconsisten and evidence-based probability*

- *If* **L** *is* **CL** *, an* **L***-probability is a classical probability function.*
- *if* **L** *is* **LJ***, an* **L***-probability is an intuitionistic probability function.*
- *if* **L** *is* **Ci***, an* **L***-probability is a paraconsistent probability function.*
- *if* **L** *is* **LETj***, an* **L***-probability is an evidence-based probability function.*

Two events α and β are said to be *independent* if $P(\alpha \wedge \beta) = P(\alpha) \cdot P(\beta)$. Two events can be independent relative to one probability measure and dependent relative to another.

Some immediate consequences of the axioms are the following:

THEOREM 3.3. *1. If* δ *is any bottom particle in* **Ci** *then* $P(\delta) = 0$*.*

2. *If* ψ *and* φ *are logically equivalent in the sense that* $\psi \vdash \varphi$ $\varphi \vdash \psi$ *, then* $P(\psi) = P(\varphi)$*.*

Proof. Immediate, in view of the axioms. □

As a consequence of the previous Theorem, $P(\alpha \wedge \neg\alpha \wedge \circ\alpha) = 0$, $P(\circ\alpha \wedge \bullet\alpha) = 0$, $P(\circ\alpha \wedge \neg \circ \alpha) = 0$ and $P(\bullet\alpha \wedge \neg \bullet \alpha) = 0$, for any probability function P.

Two sentences α and β are said to be *logically incompatible* if $\alpha, \beta \vdash \varphi$, for any φ (or equivalently, if $\alpha \wedge \beta$ act as a bottom particle). Some simple calculation rules follow:

THEOREM 3.4.

1. $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$*, if* α *and* β *are logically incompatible.*
2. $P(\circ\alpha) = 2 - (P(\alpha) + P(\neg\alpha))$
3. $P(\alpha \wedge \neg\alpha) = P(\alpha) + P(\neg\alpha) - 1$
4. $P(\sim \alpha) = 1 - P(\alpha)$
5. $P(\neg \circ \alpha) = 1 - P(\circ\alpha)$

Proof. Only items (1) and (2) will be proved (the rest is routine): (1): Since α and β are logically incompatible, $\alpha \wedge \beta$ act as a bottom particle, and the result is immediate by Theorem 3.3 and Finite Additivity.
(2): Use Finite Additivity in the sentences $\circ\alpha \vee (\alpha \wedge \neg\alpha)$ and $\circ\alpha \wedge (\alpha \wedge \neg\alpha)$. □

Probabilities are sometimes seen as generalized truth values. The so-called probabilistic semantics in this way replaces the valuations $v : L \mapsto \{0,1\}$ of classical propositional logic with the probability functions ranging on the real unit interval $[0,1]$, and valuations can be regarded as degenerate probability functions. In this sense, classical logic is to be regarded as a special case of probability logic. An analogous property holds for **Ci** and the above defined notion of paraconsistent probability measure is shown below.

Define $\Vdash_P$ as a probabilistic semantic relation whose meaning is $\Gamma \Vdash_P \varphi$ if and only if for every probability function P, if $P(\psi) = 1$ for every $\psi \in \Gamma$ then $P(\varphi) = 1$. It can be shown that **Ci** is (strongly) sound and complete with respect to such probabilistic semantics:

THEOREM 3.5.
$\Gamma \vdash \varphi$ *if and only if* $\Gamma \Vdash_P \varphi$

Proof. The left-to-right direction follows directly from the axioms of probability, namely, Tautologicity and Comparison, plus the compactness property of **Ci** proofs. For the other direction an interested reader can consult [1]. □

4 Conditional probabilities and paraconsistent updating

Perhaps the most interesting use of probability in paraconsistent logic is to come to help to the so-called Bayesian epistemology, or the formal representation of belief degrees in philosophy. The well-known Bayes rule permits one to update probabilities as new information is acquired, and, in the paraconsistent case, even when such new information involves some degree of contradictoriness.

The conditional probability of α given β, for $P(\beta) \neq 0$, is defined (as usual) as:

$$P(\alpha/\beta) = \frac{P(\alpha \wedge \beta)}{P(\beta)}$$

The traditional Bayes' Theorem for conditionalization says, for $P(\beta) \neq 0$:

$$P(\alpha/\beta) = \frac{P(\beta/\alpha) \cdot P(\alpha)}{P(\beta)}$$

As usual, $P(\alpha)$ here denotes the prior probability, i.e, is the probability of α before β has been observed. $P(\alpha/\beta)$ denotes the posterior probability, i.e., the probability of α after β is observed. $P(\beta/\alpha)$ is the likelihood, or the probability of observing β given α, and $P(\beta)$ is called the marginal likelihood or "model evidence".

A paraconsistent version of Bayes' Theorem can be set up now by making the marginal likelihood $P(\beta)$ to be analyzed in terms of $P(\alpha)$, $P(\neg\alpha)$ and $P(\alpha \wedge \neg\alpha)$ when $P(\alpha \wedge \neg\alpha) \neq 0$:

THEOREM 4.1. *Paraconsistent Bayes' Condicionalization Rule (PBCR):*

$$P(\alpha/\beta) = \frac{P(\beta/\alpha) \cdot P(\alpha)}{P(\beta/\alpha) \cdot P(\alpha) + P(\beta/\neg\alpha) \cdot P(\neg\alpha) - P(\beta/\alpha \wedge \neg\alpha) \cdot P(\alpha \wedge \neg\alpha)}$$

if $P(\alpha \wedge \neg\alpha) \neq 0$.

Proof. See [1]. □

As a slogan, (PBCR) can be summarized as saying: "Posterior probability is proportional to likelihood times prior probability, and inversely proportional to the marginal likelihood analyzed in terms of ts components". Some examples of applications studied in [1] suggest the following interpretation about Bayesian paraconsistent updates: When a test (involving contradictions) is more unreliable, paraconsistent probabilities tend to be *cautiously optimistic*, that is, values tend to expect the most favorable outcome. On the other hand, when a test (again, involving contradictions) is more reliable, paraconsistent probabilities tend to be *cautiously pessimistic*, in the sense of favoring expectation of undesirable outcomes, Notwithstanding, the test is cautious in all cases.

5 Evidence and probability: the logic LETj

The logic **LETj**, introduced in [5], is a paraconsistent and intuitionistic Logic of Evidence and Truth, whose main intuitive motivations are the following:

- 'A holds' means 'there is evidence that A is true';
- 'A does not hold' means 'there is no evidence that$A$ is true';
- '$\neg A$holds' means 'there is evidence that A is false';
- '$\neg A$ does not hold' means 'there is no evidence that A is false'.

It is to be remarked that neither excluded middle nor explosion hold in **LETj** because evidence can be *incomplete* as well as *contradictory* (albeit reasoning about evidence does not need to be *explosive*.

The axiomatic presentation of **LETj** is the following:

DEFINITION 5.1. *The system* ***LETj*** *is composed by:*

1. *Axioms*
   (a) *(**PC**$^+$) all positive axioms of **PC**, minus $p \vee (p \supset q)$ ("Dummett's law").*
   (b) *(**bC1, or Gentle Explosion**)* $\circ p \supset [p \supset (\neg p \supset q)]$
   (c) *(**Classicality**)* $\circ p \supset (p \vee \neg p)$
   (d) *(**DN**)* $\neg\neg p \equiv p$
   (e) *($\supset$-De Morgan)* $\neg(p \supset q) \equiv p \wedge \neg q$
   (f) *($\vee$-De Morgan)* $\neg(p \vee q) \equiv \neg p \wedge \neg q$
   (g) *($\wedge$-De Morgan)* $\neg(p \wedge q) \equiv \neg p \vee \neg q$
2. *Rule of MP*

Notice that $p \vee \neg p$ does not hold and new axioms have been introduced in comparison with the purely paraconsistent logic **Ci**. Distinct negations can be defined in **LETj**, besides $\neg$:

- An intuitionistic negation: $\sim \alpha \overset{def}{=} \alpha \supset [p \wedge (\neg p \wedge \circ p)]$

- A strong negation: $\simeq \alpha \stackrel{def}{=} \neg\alpha \ \wedge \circ\alpha$

Though they are equivalent in **Ci**, they do not coincide in **LETj**

- $\sim \alpha \nvdash \simeq \alpha, \qquad \simeq \alpha \vdash \sim \alpha$
- $\nvdash (\alpha \vee \sim \alpha \vee \circ\alpha) \qquad \nvdash (\alpha \vee \simeq \alpha \vee \circ\alpha)$
- $\alpha \vee \sim \alpha \nvdash \alpha \vee \simeq \alpha, \qquad \alpha \vee \simeq \alpha \nvdash \alpha \vee \sim \alpha$

Some nice logical features of **LETj** are the following (for proofs, the reader is invited to see [5]):

- $\neg(\alpha \wedge \neg\alpha)$ and $(\alpha \vee \neg\alpha)$ are logically equivalent in **LETj**, but neither of them is logically equivalent to $\circ\alpha$, i.e. non-contradiction and excluded middle coincide, but differ from classicality
- It is impossible to prove classicality: **LETj** has no theorems of the form $\circ\alpha$
- Classicality propagates: if $\circ\alpha_1, \circ\alpha_2 \ldots \circ\alpha_n$ hold, any formula depending only on $\alpha_1, \alpha_2, \ldots, \alpha_n$ formed by $\supset$, $\wedge$, $\vee$ and $\neg$ behave classically

The notion of probability defined in **LETj** is of particular interest, since it defines probability functions which are closed to a measure of evidence. Some features of the probability functions over **LETj** are, for α and β formulas in **LETj**:

1. $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$, if α and β are logically incompatible.
2. $P(\simeq \alpha) = 1 - P(\alpha)$
3. $P(\sim \alpha) \neq 1 - P(\alpha)$ in general
4. $P(\circ\alpha \vee \neg(\alpha \vee \neg\alpha)) = P(\circ\alpha) + P(\neg(\alpha \vee \neg\alpha))$
5. $P(\circ\alpha) \leq P(\alpha \vee \neg\alpha) \leq 1$
6. $P(\circ\alpha) \leq P(\alpha \supset (\neg\alpha \supset \beta)_{\leq} 1$

To sum up, evidence and truth in **LETj** are related in a paraconsistent and paracomplete way, taking into account that the following possible scenarios are possible for a certain situation:

- No evidence at all: both α and $\neg\alpha$ do not hold.
- Only evidence that α is true: α holds, $\neg\alpha$ does not hold.
- Only evidence that α is false: $\neg\alpha$ holds, α does not hold.
- Conflicting evidence: both α and $\neg\alpha$ hold.

LETj is designed to express the notions of conclusive and non-conclusive evidence, as well as the preservation of evidence; it is also able to recover classical logic for propositions whose truth-value have been conclusively established. In this way, it can also express the notion of preservation of truth. Consequently, a notion of probability based on **LETj** will reflect such scenarios with exciting possibilities and unfoldings.

6 Summary, comments and conclusions

I have reviewed some basic points about paraconsistency, characterizing a paraconsistent logic, in general terms, as a logical system endowed with a notion of consistency $\circ$ and a negation $\neg$ which is free from trivialism, in the sense that a contradiction expressed by means of a negation $\neg$ does not necessarily trivialize the underlying consequence relation, although consistent contradictions do explode. A measure of probability has been defined for two such logics, for the system **Ci**and for the system **LETj**, taking profit from the underlying notion of consistency, and essaying the first steps towards new versions of paraconsistent Bayesian updating.

How can paraconsistent probabilities be interpreted? One possible viewpoint is to interpret paraconsistent probabilities as degrees of belief that a rational agent attaches to events, in such a way are such degrees respect the following principles: the necessary events (for instance, tautologies) get maximum degrees, impossible events (for instance, bottom particles) get lowest degrees, probabilities respect logical consequence, and finite additivity is guaranteed. The last condition seems to be less obvious, but the so-called Dutch Book arguments provide, at least for the classical case, a line of justification for keeping finite additivity. It should be taken into account that our underlying logic **Ci** *enlarges* the classical scenarios in important ways: so for instance, even if impossible events should have degree zero by a rational agent, neither events of degree zero by a rational agent are necessarily impossible, nor a contradiction is an impossible event (although a consistent contradiction is, as commented above).

The core question, however, is not whether the laws of probability should be classified as laws of logic, but how logic and probability could be combined to refine reasoning. In this respect, considering that probability theory differs from classical logic in various aspects, and paraconsistent logic differs as well, and that both are tolerant to contradictions, inexactness, and so on, their combination offers a new and exciting reasoning paradigm.

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References

1. J. Bueno-Soler and W. Carnielli. May be and may be not: paraconsistent probability from the lfi viewpoint. *CLE e-prints*, 15(2), 2015. `ftp://ftp.cle.unicamp.br/pub/e-prints/vol.15,n2,2015.pdf`

2. J. Bueno-Soler and W. A. Carnielli. Experimenting with consistency. Experimenting with consistency. In V. Markin and D. Zaitsev, editors, *The Logical Legacy of Nikolai Vasiliev and Modern Logic*. Synthese Library — Studies in Epistemology, Logic, Methodology, and Philosophy of Science-Springer, 2015. Preprint available at: `http://www.cle.unicamp.br/e-prints/vol_14,n_3,2014.html`

3. W. A. Carnielli, M. E. Coniglio, and J. Marcos. Logics of formal inconsistency. In D. Gabbay and F. Guenthner, editors, Handbook of Philosophical Logic, volume 14, pages 1–93, Amsterdam, 2007. Springer-Verlag

4. W. Carnielli, M.E. Coniglio, R. Podiacki, and T. Rodrgues. On the way to a wider model theory: Completeness theorems for first-order logics of formal inconsistency. *Review of Symbolic Logic*, 3:548–578, 2015

5. W. Carnielli and A. Rodrigues. A logic for evidence and truth. *CLE e-prints*, 15(5), 2015. `ftp://ftp.cle.unicamp.br/pub/e-prints/vol.15,n_5,2015.pdf`

6. E. Mares. Paraconsistent probability theory and paraconsistent Bayesianism. *Logique et Analyse*, 160:375–384, 1997

7. G. Priest. *In Contradiction: A Study of the Transconsistent*. Martinus Nijhoff, Amsterdam, 1987

8. L. Roberts. Maybe, maybe not: Probabilistic semantics for two paraconsistent logics. In D. Batens, C. Mortensen, G. Priest, and J. P van Bendegem, editors, *Frontiers of Paraconsistent Logic: Proceedings of the I World Congress on Paraconsistency*, Logic and Computation Series, pages 233–254. Baldock: Research Studies Press, King's College Publications, 2000

9. J. N. Williams. Inconsistency and contradiction. *Mind*, 90:600–602, 1981