The 1900 Turn in Bertrand Russell's Philosophy of Mathematics and the Emergence of his Paradox

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Summary. Russell's main academic project was to make mathematics rigorous, reducing it to logic. Before August-1900, Russell's logic was based on mereology. His acquaintance with Peano's ideas in August-1900, however, led him to discard the part–whole logic and to adopt a kind of intensional logic instead. Among other things, the predicate logic helped Russell to embrace a technique of treating infinite collections with the help of a singular concept which he called 'denoting phrase'. He hoped that this step will give him the resources for a paradox-free treating of infinity number. Unfortunately, this hope remained unfulfilled; his new conception only removed the paradoxon of infinity from the realm of infinite classes (ordinal numbers) to that of class-inclusion (cardinal numbers).
Russell's long-elaborated solution to his paradox developed between 1905 and 1908 was nothing but setting aside some of the ideas he adopted with his August-1900 turn: (i) With Theory of Descriptions, Russell reintroduced elements of reality (complexes we are acumainted with) into logic. In this way he partly restored the pre-August 1900 realistic mereology of complexes and simples. (ii) The elimination of classes with the help of the so-called "substitutional theory", and (iii) the elimination of propositions by means of the multiple relation theory of judgment completed this process.

Russell as a Mereologist. Russell's short period as a Neo-Hegelian in philosophy of mathematics was replaced by, what can be called, his analytic philosophy of mathematics substaintiated by the logic of relations. Russell made his way in direction of analysis after he read A. N. Whitehead's A Treatise on Universal Algebra in March 1898. At that time he maintained that mathematics starts with simple, indefinable fundamental ideas, from which its body is build out through simple, indefinable axioms.
In contrast to his philosophy of mathematics, Russell's relational logic became predominantly analytical only after he read Cantor's Grundlagen einer allgemeinen Mannigfaltigkeitslehre at the beginning of July 1899. The change is documented in 'Fundamental Ideas and Axioms of Mathematics', where Russell adopted a full-blooded mereology for the first time. Now he clamined that central in logic (not in ontology) is the relation of 'logical priority', i.e. the relation of whole and part.
In Russell's part–whole logic the logical consequence holds between both terms and propositions. So in 'Fundamental Ideas and Axioms of Mathematics' it is assumed that 'it is possible for simple concepts [too, i.e., not only for propositions] to imply others' (1899c, p. 293). At that point in time, central place in Russell's logic played, what was later called, the relation of "ontological dependence".

The Turn. In the first years of the new millennium, Russell gradually adopted two novelties in his logic that were developed in full first in the *Principles of Mathematics*: material implication and predicate logic (theory of denoting).

First Symptom that the Turn was not Advantageous: the Paradoxes. So far I have found that in an attempt to escape from the paradox of infinity, in The Principles of Mathematics Russell accepted class-concept and thing (individual) as radically opposite terms. In this he followed the new many-order logic of Peano (and later that of Frege), which embraced an opposition between class-concepts, on the one hand, and individuals and terms, on the other, etc.
Unfortunately, as a by-product of this treatment, another paradox emerged—the paradox of classes. All this suggests, and I will examine this point in a while, that Russell's Peano–Frege

turn did not eliminate the paradox of infinity—it merely removed it from one realm into another; that is, from the realm of infinite classes, to that of class-inclusion.

This point did not remain unnoticed by commentators. According to one of them, Russell's official theory was that mathematics is free from paradoxes. Deep in mind, however, he continued to believe that mathematics is paradoxical (Garciadiego 1992, p. 152). Another scholar has noted that all the three paradoxes which Russell tried to resolve—(i) the paradox of infinite ordinal number (discovered July 1899), (ii) the paradox of the largest cardinal number (discovered November 1900), and (iii) Russell's paradox proper (discovered May 1901)—have one and the same structure. Apparently, 'this structure was presented in the back of his mind as a kind of template that could be unconsciously applied to Cantor's work on infinite number.' (Moore 1995, p. 236) In what follows in this paper I shall try to describe this template precisely.

The Way Out—Theory of Descriptions and other Emendations. The Theory of Descriptions was introduced mainly in order to repair these two confusions. As a matter of fact, Russell felt that the problems of paradoxicality can be resolved through a correct theory of descriptions from the very beginning. Later he remembered that after years of abortive efforts to solve the paradoxes, the first success was the Theory of Descriptions. 'This was, apparently, not connected with the contradictions, but in time an unsuspected connection emerged.' (1959, p. 79)

But what exactly was the connection between the solution of the paradoxes of self-reference and the Theory of Descriptions? Above all, with the acceptance of the objects of acquaintance as a legitimate part of logic, Russell (re)introduced elements of reality—complexes we are aquainted with—into logic. In this way, Theory of Descriptions limited the competence of apophantism, and thus partly restored the realistic mereology of complexes and simples, embraced in 1898, but rejected in 1900.

Russell also made two other emendations to his Peao–Fregean logic both of which were directed at eliminating the splitting of logic into different orders:

(i) Russel first eliminated classes. After 1905 he maintained that classes are only "incomplete symbols" and so are not objects. There are no classes but propositions and propositional functions. According to the Ramified Theory of Types, there are also types of classes (attributes), but these are only one type of variables. In other words, there is a hierarchy of classes but not of objects.

(ii) Unfortunately, two years later, in 1907, Russell discovered that propositions produce paradoxes of their own. (Of course they do: they are only unities.) In consequence, he came to maintain that there are no propositions. To be more exact, propositions were eliminated with the help of the multiple relation theory of judgment.

References

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