A new reading and comparative interpretation of Gödel's completeness (1930) and incompleteness (1931) theorems

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Prehistory and background

There are some statements referring the exceptionally famous papers of Gödel [1-2] and their interpretation, which turn out to be rather misleading. Their essence consists in linking (the logical) completeness [1] to finiteness, and correspondingly (the arithmetical or equivalent) incompleteness (or alternatively, inconsistency, [2]) to infinity.

Their historical background and problematics have been reconstructed nowadays by means of (i) the crisis in the foundation of mathematics, being due to actual infinity in the "naïve" set theory, (ii) its axiomatiozations, reducing the problem of their completeness and consistence to that of their models in Peano arithmetic, (iii) Hilbert's program for the arithmetical foundation of mathematic, and (iv) Russell's construction in *Principia*.

(v) Skolem's conception (called also paradox) about the "relativity of the concept of set" [3], once the axiom of choice is utilized, should be specially added to that background to be founded the present viewpoint.

A sketch of the present viewpoint

(S1) One can trivially demonstrate that *Peano arithmetic excludes infinity* from its scope fundamentally: Indeed, 1 is finite; adding 1 to any natural number, one obtains a finite natural number again; consequently, all natural numbers are finite according to the axiom of induction.

(S2) Utilizing the axiom of choice equivalent to the well-ordering principle (theorem), any set can be one-to-one mapped in some subset of the natural numbers. As Skolem emphasized expressively, this means that any set even being infinite (in the sense of set theory) *admits* an ("nonintrinsic" or "unproper") one-to-one model by some subset of the natural numbers, which should be finite, rather than only by a countably infinite model.

(S3) In fact, the so-called countable power of a set is introduced in the (Cantor, or "naïve") set theory as the power equivalent to that of all natural numbers and different (and bigger) than that of any finite number. However, *the number of all natural numbers should be a natural number and thus finite* in Peano arithmetic as a corollary from (1) above.

(S4) Consequently, if one compares Peano arithmetic and set theory (e.g. in ZFC axiomatization), a discrepancy about (countable) infinity is notable:

(S4.1) *Peano arithmetic is incomplete to set theory* for that arithmetic does not contain any infinity (including the countable one).

(S4.2) Furthermore, *Peano arithmetic cannot be complemented* by any "axiom of infinity" because it contains only finite numbers according (1) above. In other words, if it is complemented to become "complete" in the sense of S4.1, it would become inconsistent furthermore. Those statements (S4.1−S4.2) reconstruct Gödel's incompleteness [2] argument in essence, but in a trivial way.

(S5) If one considers a logical axiomatization in the sense of *Principia* (as in [1]) without any mapping and even correspondence to Peano arithmetic, some axiom of infinity is implicitly allowed, and thus completeness provable, but only nonconstrucively for whether explicit or implicit reference to infinity does not admit any constructiveness in a constructive way in principle. (One can mean some constructiveness in a nonconstructive way, i.e. as some constructiveness of "pure existing" by virtue of the axiom of choice, e.g. as the fundamentally random choice of some finite set to represent a given infinite set for Skolem's relativeness of 'set': this involves *probability theory in the foundation of mathematics*.)

Thesis

(T1) *Peano arithmetic cannot serve as the ground of mathematics* for it is inconsistent to infinity, and infinity is necessary for its foundation. Though Peano arithmetic cannot be complemented by any axiom of infinity, there exists at least one (logical) axiomatics consistent to infinity. That is nothing else than right a new reading at issue and comparative interpretation of Gödel's papers meant here.

(T2) Peano arithmetic *admits* anyway generalizations consistent to infinity and thus to some addable axiom(s) of infinity. The most utilized example of those generalizations is the separable complex Hilbert space.

(T3) Any generalization of Peano arithmetic consistent to infinity, e.g. the separable complex Hilbert space, can serve as a *foundation for mathematics* to found itself and by itself.

A few main arguments

(A1) Skolem's relativeness of 'set'

(A2) The viewpoint to Gödel's papers sketched above (in S1–S5).

(A3) The separable complex Hilbert space can be considered as a *generalization of Peano arithmetic* as follows. Hilbert space is an infinite series of qubits. A qubit is defined as usual and thus isomorphic to a unit ball in which two points are chosen: the one from the ball, the other from its surface. Any point in that space would representable as some choice (record) of values in each qubit. If the radiuses of all those unit balls are degenerate to 0, the complex Hilbert space is reduced to Peano arithmetic. On the contrary, if two choices, each one among a limited uncountable set and thus representable as a normed pair of complex numbers, are juxtaposed to any natural number, one obtains the complex Hilbert space as a series of qubits and as a generalization of Peano arithmetic. The essential property of the separable complex Hilbert space (together with its dual space) as that model is that the set of all natural numbers is mapped one-to-one to a series of infinite sets (which is identically doubled). Thus the set of all natural numbers is representable as a series of bits, e.g. the "tape" of Turing's machine, and as a single qubit, e.g. a "cell" of the *quantum* Turing machine.

(A4) The *theorems of the absence of hidden variables* in quantum mechanics [4-5] can be interpreted as a completeness proof of the above model based on the separable complex Hilbert space. Indeed, the separable complex Hilbert space is sufficient for the proof of those theorems, and the absence of hidden variables corresponds unambiguously to completeness. Any hidden variable would mean the incompleteness of the separable complex Hilbert space.

References

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