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A MULTIPLE WORLD PARADOX OF THE CONCURRENT DYNAMIC LOGIC [*]

Abstract: A multiple world paradox of the Concurrent Dynamic Logic is introduced. It may limit the implementation field of the Concurrent Dynamic Logic for the reasoning of the programs. The reasoning might be restricted, at least up to the independent atomic programs which do not form interfering concurrent compound processes.
Keywords: Multiple world paradox, Concurrent Dynamic Logic, interactive processes, parallel processes.

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ПАРАДОКС КРАТНЫХ МИРОВ КОНКУРЕНТНОЙ ДИНАМИЧЕСКОЙ ЛОГИКИ

Аннотация: Сформулирован парадокс кратных миров Конкурентной динамической логики. Он может несколько ограничить область применения Конкурентной динамической логики в качестве средства рассуждения о выполнении программ, которое может свестись, по крайней мере, до независимых атомарных программ, которые не образуют взаимодействующие параллельные сложные процессы.

Ключевые слова: Парадокс кратных миров, конкурентная динамическая логика, взаимодействующие процессы, параллельные вычисления.

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1 Introduction

There are many examples of defining the parallel processes in mathematics (see e.g.[2]), in theoretical computer science (see e.g.[3]), in theory of formal languages (see e.g.[4], [8], [9], [11], [12]). The notion of multiple states is successfully incorporated in different ways in most of the formal systems.

In last decades, logics involved parallel processes in its description as well (see e.g. [1], [5], [7], [10]). However, the usage of the notion of multiple world in logics can meet some restrictions. One of it is discussed in the paper.

2 Concurrent Propositional Dynamic Logic

The *Concurrent Propositional Dynamic Logic* (CPDL)[10] is an extension of the *Propositional Dynamic Logic* (PDL)[6]. CPDL was developed for reasoning about the parallel programs. It uses the familiar syntax of the PDL, augmented with the *concurrency operator* $\cap$.

Definition 2.1 The syntax of Concurrent Dynamic Logic *is given by the sets of programs* Π *and formulas* Φ*. The sets are composed by the propositional atomic formulas* ϕ_i *from the set of propositional atomic formulas* Φ_0*, and the atomic programs* π_i *from the set of atomic programs* Π_0 *according to the following rules of generation:*

1. $\Phi_0 \subseteq \Phi$
2. *if* φ*,* $\psi \in \Phi$*, and* $\alpha \in \Pi$*, then* $\varphi \vee \psi$*,* $\neg\varphi$*,* $\langle\alpha\rangle\varphi$*,* $[\alpha]\varphi \in \Phi$
3. $\Pi_0 \in \Pi$
4. *if* α*,* $\beta \in \Pi$*, and* $\varphi \in \Phi$*, then* $\alpha \cup \beta$*,* $\alpha \cap \beta$*,* $\alpha;\beta$*,* α^**,* $\varphi? \in \Pi$

The interpretation of formulas and programs in CPDL semantics for the model $\langle \mathcal{W}, \{\mathcal{R}_{\pi_i}\}_{\pi_i \in \Pi_0}, \mathcal{V}\rangle$ is considerably different from their interpretation for PDL. Both for CPDL and PDL models, a set of possible worlds $\mathcal{W}$ is interpreted as a set of all possible states of the programs. The worlds of the model together with the accessibility relations determine a directed graph. The accessibility relations $\{\mathcal{R}_{\pi_i}\}_{\pi_i \in \Pi_0}$ are interpreted as executions of atomic programs, connecting the income and the outcome worlds. A valuation function $\mathcal{V}$ maps each atomic formula onto a subset of all possible worlds. This mapping is interpreted as an assignment of the truth values to the atomic formulas, individually for each given world.

In PDL, however, $\{\mathcal{R}_{\pi}\}_{\pi_i \in \Pi_0}$ are defined on a subset $\mathcal{W} \times \mathcal{W}$. In this case $(w, w') \in \mathcal{R}_\pi$ means that program π can be executed when the conditions of the possible world w are met and then it will halt in the world w'.

On the other hand, in CPDL, the accessibility relations are defined on $\mathcal{W} \times 2^{\mathcal{W}}$. Thus, the relations contain pairs (w, U), where $w \in \mathcal{W}$ and $U \subseteq \mathcal{W}$. This is done so to make it possible to express the programs starting in some world w and simultaneously reaching all worlds of U after execution. Hence, a formula $\langle\pi\rangle\varphi$ in CPDL is being executed in a world w if and only if there exists a set $U \subseteq \mathcal{W}$ such that $(w, U) \in \mathcal{R}_\pi$ and formula φ is true in each world $w' \in U$. According to that the CPDL model is defined as follows:

Definition 2.2 A model of Concurrent Propositional Dynamic Logic $\mathcal{M}$ *is a tuple* $\langle \mathcal{W}, \{\mathcal{R}_{\pi_i}\}_{\pi_i \in \Pi_0}, \mathcal{V}\rangle$, *where*

1. $\mathcal{W}$ *is a nonempty set of the states of the programs*
2. $\mathcal{R}_{\pi_i} \subseteq \mathcal{W} \times 2^{\mathcal{W}}$ *is the accessibility relation describing the execution of the atomic program* $\pi_i \in \Pi_0$. *The rules below extend the definition of the accessibility relation onto the entire set* Π *of programs. Let* $\alpha, \beta \in \Pi$, $\varphi \in \Phi$, *where* Φ *is the set of formulas. Suppose that* $w \in \mathcal{W}$, *and* $S, T, U \subseteq \mathcal{W}$ *then*

$$\begin{aligned}
\mathcal{R}_{\varphi?} &= \{(w, \{w\}) \mid w \in \mathcal{V}(\varphi)\}\\
\mathcal{R}_{\alpha\cup\beta} &= \mathcal{R}_\alpha \cup \mathcal{R}_\beta\\
\mathcal{R}_{\alpha\cap\beta} &= \{(w, \{U\}) \mid \exists S, T((w, S) \in \mathcal{R}_\alpha, (w, T) \in \mathcal{R}_\beta,\\
&\quad U = S \cup T)\}\\
\mathcal{R}_{\alpha;\beta} &= \{(w, \{U\}) \mid \exists S \subseteq \mathcal{W}((w, S) \in \mathcal{R}_\alpha \wedge\\
&\quad \forall s \in S \exists U_s \subseteq U((s, U_s) \in \mathcal{R}_\beta \wedge U = \bigcup_{s\in S} U_s))\}\\
\mathcal{R}_{\alpha^*} &= \{(w, \{U\}) \mid w \in T,\ \mu T.(U \subseteq T, \mathcal{R}_\alpha = (t, T) \mid t \in T)\}
\end{aligned}$$

3. $\mathcal{V} \subseteq \Phi \times 2^{\mathcal{W}}$ *is the valuation function, for every atomic formula* $\phi_i \in \Phi_0$ *assigning one of the truth values* 0 *or* 1, *separately for each state* $w \in \mathcal{W}$. *The rules below extend the definition of the valuation function to the set* Φ *of formulas. Let* $\varphi, \psi \in \Phi$, $\alpha \in \Pi$.

$$\begin{aligned}
\mathcal{V}(\varphi \vee \psi) &= \mathcal{V}(\varphi) \cup \mathcal{V}(\psi)\\
\mathcal{V}(\neg\varphi) &= W - \mathcal{V}(\varphi)\\
\mathcal{V}(\langle\alpha\rangle\varphi) &= \{w \mid \exists U((w, U) \in \mathcal{R}_\alpha \wedge U \subseteq \mathcal{V}(\varphi))\}\\
\mathcal{V}([\alpha]\varphi) &= \{w \mid \forall U((w, U) \in \mathcal{R}_\alpha \Rightarrow U \subseteq \mathcal{V}(\varphi))\}
\end{aligned}$$

The other connectives are defined via the given ones, in a usual way for the PDL.

Definition 2.3 *The given model of CPDL is sound and complete with regard to the following* system of axioms and inference rules[1]
Axiom schemes

A1 All tautologies of the propositional calculus

[1] Proven by D. Peleg (see [10])

A2 $\langle\alpha;\beta\rangle\varphi \equiv \langle\alpha\rangle\langle\beta\rangle\varphi$

A3 $\langle\alpha\cup\beta\rangle\varphi \equiv \langle\alpha\rangle\varphi \vee \langle\beta\rangle\varphi$

A4 $\langle\alpha\cap\beta\rangle\varphi \equiv \langle\alpha\rangle\varphi \wedge \langle\beta\rangle\varphi$

A5 $\langle\alpha^*\rangle\varphi \equiv \varphi \vee \langle\alpha\rangle\langle\alpha^*\rangle\varphi$

A6 $\langle\psi?\rangle\varphi \equiv \psi \wedge \varphi$

The inference rules are:

$$\frac{\varphi, \varphi\supset\psi}{\psi} \qquad \frac{\varphi\supset\psi}{\langle\alpha\rangle\varphi\supset\langle\alpha\rangle\psi} \qquad \frac{\langle\alpha\rangle\varphi\supset\varphi}{\langle\alpha^*\rangle\varphi\supset\varphi}$$

3 A Multiple World Paradox

Consider atomic programs $\pi_1, \pi_2 \in \Pi_0$, and atomic formulas $\phi_1, \dots, \phi_6 \in \Phi_0$, and assume that for the model $\mathcal{M}_{ex} = \langle\{w_1, \dots, w_5\}, \{\mathcal{R}_{\pi_1}, \mathcal{R}_{\pi_2}\}, \mathcal{V}\rangle$ of the CPDL

$(w_1, \{w_2\}), (w_3, \{w_4\}) \in \mathcal{R}_{\pi_1}$,
$(w_1, \{w_3\}), (w_2, \{w_4\}) \in \mathcal{R}_{\pi_2}$,
$w_1, w_3 \in \mathcal{V}(\phi_1)$
$w_1, w_2 \in \mathcal{V}(\phi_2)$
$w_2, w_4 \in \mathcal{V}(\phi_3)$
$w_3, w_4 \in \mathcal{V}(\phi_4)$
$w_5 \in \mathcal{V}(\phi_5), \mathcal{V}(\phi_6)$

Particularly, this model corresponds to the cases when a permutation of two *independent programs* is examined. Then both sequential executions $\pi_1;\pi_2$ and $\pi_2;\pi_1$ of the programs will bring the same result. In the given model, $\mathcal{R}_{\pi_1;\pi_2}$ and $\mathcal{R}_{\pi_2;\pi_1}$ lead to the same world w_4 from the initial world w_1, i.e.

$(w_1, \{w_4\}) \in \mathcal{R}_{\pi_1;\pi_2}$, and $(w_1, \{w_4\}) \in \mathcal{R}_{\pi_2;\pi_1}$.

Then, based on the definition of the model of CPDL, one may infer that

$$(w_1, \{w_4\}) \in \mathcal{R}_{\pi_1;\pi_2\cap\pi_2;\pi_1} \tag{1}$$

In spite of the fact that we involve the independent programs, regarding the concurrency operator we make an interfering construction from them. This allow us to illustrate a multiple world paradox of the Concurrent Dynamic Logic.

Consider the following interpretation of the involved atomic programs and the propositional atomic variables.

π_1 ::= "$\texttt{Reg}_1$ $++$"
π_2 ::= "$\texttt{Reg}_2$ $++$"
ϕ_1 ::= "$\texttt{Reg}_1$ is 0"
ϕ_2 ::= "$\texttt{Reg}_2$ is 0"

ϕ_3 ::= “$\texttt{Reg}_1$ is 1”
ϕ_4 ::= “$\texttt{Reg}_2$ is 1”
ϕ_5 ::= “$\texttt{Reg}_1$ is 2”
ϕ_6 ::= “$\texttt{Reg}_2$ is 2”
According to this example we get that
$(w_1, \{w_4\}) \in \mathcal{R}_{\pi_1;\pi_2}$, and $(w_1, \{w_4\}) \in \mathcal{R}_{\pi_2;\pi_1}$,
whereas

$$(w_1, \{w_5\}) \in \mathcal{R}_{\pi_1;\pi_2 \cap \pi_2;\pi_1}$$

and

$$(w_1, \{w_4\}) \notin \mathcal{R}_{\pi_1;\pi_2 \cap \pi_2;\pi_1} \quad (2)$$

that contradicts with the property (1) of the model. In terms of the axiomatics this means that

$$\mathcal{M}_{ex}, w_1 \models \langle \alpha; \beta \rangle (\phi_3 \wedge \phi_4) \quad (3)$$
$$\mathcal{M}_{ex}, w_1 \models \langle \beta; \alpha \rangle (\phi_3 \wedge \phi_4) \quad (4)$$
$$\mathcal{M}_{ex}, w_1 \models \langle \alpha; \beta \cap \beta; \alpha \rangle (\phi_5 \wedge \phi_6) \quad (5)$$

and

$$\mathcal{M}_{ex}, w_1 \models \langle \alpha; \beta \cap \beta; \alpha \rangle (\neg\phi_3 \wedge \neg\phi_4) \quad (6)$$

that is

$$\mathcal{M}_{ex}, w_1 \models \langle \alpha; \beta \cap \beta; \alpha \rangle (\neg\phi_3 \vee \neg\phi_4) \quad (7)$$

and

$$\mathcal{M}_{ex}, w_1 \models \langle \alpha; \beta \cap \beta; \alpha \rangle \neg(\phi_3 \wedge \phi_4) \quad (8)$$

Thus, bringing together (3), (4), and (8), we obtain that the axiom *A4* will not be valid for the given interpretation.

4 Conclusion

We obtain that interpretations of the Concurrent Dynamic Logic may easily lead to the multiple world paradox. It is clear that the paradox is connected with interdependency of the programs involved into the parallel execution. If these programs use common resources, then their concurrent execution may give a result that cannot be obtained straightforward from the results of their nonconcurrent executions. Thus the description of the behaviour of the concurrent programs involving the multiple states may not be plausible in logics.

However, what is important is that the paradox is obtained even for the independent atomic programs, by interfering them in the compound concurrent program in a special way.

Thus, one may avoid the paradox while using the Concurrent Dynamic Logic by applying quite a strong restriction. Besides that the programs should be independent, they cannot permit interfering in scope of the concurrency operator.

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