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KANTIANS MOTIVES IN INTUITIONISTIC LOGIC

J. Hintikka notes that “what is needed for the logical necessity of a sentence p in a world w_0 is more than its truth in each one of the arbitrary selected set of alternatives to w_0. What is needed is its truth in each logically possible world” [1, p.90]. Yet nothing prompts us whether all such worlds belong to alternatives of given world. Maybe the way out would be replacing logical (metaphysical according to Kripke) modalities with some suitable variant of transcendental modalities.

This proposal was realized in [3] by appealing to Kant’s conception of transcendentality (“we can cognize of things a priori only what we ourselves have put into them” [5, p.111]), introducing a family of binary (accessibility) relations on a set of possible worlds and combining respective modal logic with L. Humberstone’s inaccessibility logic [2]. It seems that this approach will be pertinent for intuitionistic logic too especially in the framework of Kripke semantics generalized by van Dalen [4].

Such semantics is based on the conception of mathematics (and hence logic) as a mental activity of a mathematician (or logician respectively) S. His mental activity is structured in linear time (runs through 0,2,3, …) where at each time S has acquired a certain knowledge which increases monotone in time. This process should be presented as treelike picture of S’s possible histories. Each node of the tree represents a stage of knowledge of S and we have assigned a set of sentences S_i subject to the condition that S_i increase.

More formally, let us consider the usual model which is a triple $\mathbf{M} = \langle M, \leq, \Vdash\rangle$ where M is partially ordered by $\leq$ and the relation $\Vdash$ between elements of M and sentences, called the *forcing relation*, is inductively defined. Firstly, we replace $\leq$ with the family $\{\leq_i\}_{i\in I}$ and define the following conditions:

$a\Vdash\varphi\supset\psi$ if $\forall i\in I\ \forall b\geq_i a\ (b\Vdash\varphi\Rightarrow b\Vdash\psi)$

$a\Vdash\varphi\supset_i\psi$ if $\exists i\in I\ \forall b\geq_i a\ (b\Vdash\varphi\Rightarrow b\Vdash\psi)$

$a\Vdash\neg\varphi$ if $\forall i\in I\ \forall b\geq_i a\ (b\nVdash\varphi)$

$a\Vdash\neg_i\varphi$ if $\exists i\in I\ \forall b\geq_i a\ (b\nVdash\varphi)$

We can introduce the transitive closure of all relations $\geq^o = (\geq_1\cup\geq_2\cup\ldots\cup\geq_n)$ (supposing $I=\{1, 2, \ldots, n\}$) and rewrite the truth-condition for $\Vdash\varphi\supset\psi$ and $\neg\varphi$ as

$a\Vdash\varphi\supset\psi$ if $\forall b\geq^o a\ (b\Vdash\varphi\Rightarrow b\Vdash\psi)$

$a\Vdash\neg\varphi$ if $\forall b\geq^o a\ (b\nVdash\varphi)$

Here $\supset$ stands for “logical” implication, $\supset$ for implication for agent i, $\neg$ for “logical” negation and $\neg_i$ for negation for agent i. It is obvious that we will have $(\varphi\supset\psi)\supset(\varphi\supset_i\psi)$ and $\neg\varphi\supset\neg_i\varphi$.

Now we enrich our language with the operator K and K_i where $\mathrm{K}\varphi$ means “Everyone knows, that φ” (common knowledge), and $\mathrm{K}_i\varphi$ means “An agent i knows, that φ” (individual knowledge). Then we add two conditions:

$a\Vdash\mathrm{K}\varphi$ iff $\forall b\geq^o a\ (b\Vdash\varphi)$

$a\Vdash\mathrm{K}_i\varphi$ iff $\exists i\in I\ \forall b\geq_i a\ (b\Vdash\varphi)$

Defining dual operators for K and K_i with the help of definitions

$a\Vdash\mathrm{B}\varphi$ iff $\exists b\geq^o a\ (b\Vdash\varphi)$

$a\Vdash\mathrm{B}_i\varphi$ iff $\exists i\in I\ \exists b\geq_i a\ (b\Vdash\varphi)$

we would be able to read them as follows: $\mathrm{B}\varphi$ means “Everyone believes that φ” (common belief), $\mathrm{B}_i\varphi$ means “An agent i believes that φ” (individual belief). Among axiom schemes for such operators will be $\mathrm{K}\varphi\supset\mathrm{B}\varphi$, $\mathrm{K}_i\varphi\supset\mathrm{B}_i\varphi$.

The next step is to define some new conditions based on the “inaccessibility” relation:

$a\Vdash\varphi\Supset\psi$ if $\forall b\ngeq^o a\ (b\Vdash\varphi\Rightarrow b\Vdash\psi)$

$a\Vdash\varphi\Supset_i\psi$ if $\exists i\in I\ \forall b\ngeq_i a\ (b\Vdash\varphi\Rightarrow b\Vdash\psi)$

$a \Vdash \sim\varphi$ if $\forall b \ngeq^{o} a\ (b \nVdash \varphi)$

$a \Vdash \sim_i \varphi$ if $\exists i \in I\ \forall b \ngeq_i a\ (b \nVdash \varphi)$

where $\varphi \Supset \psi$ means φ "transcendently" (beyond the limits of all possible experience and knowledge) implies ψ, $\varphi \Supset_i \psi$ means φ "transcendently" implies ψ for agent *i*, $\sim\varphi$ is a "transcendent" negation of φ and $\sim_i\varphi$ is a "transcendent" negation of φ for agent i. Again, as for $\supset$, $\supset_i$ and $\neg$, $\neg_i$ we will have $(\varphi \Supset \psi) \supset (\varphi \supset_i \psi)$ and $\sim\varphi \supset \sim_i \varphi$.

Pursuing an analogy with transcendental modalities in [3] (the possible worlds conform to the mind: in knowing, it is not the mind that conforms to possible worlds but instead possible worlds that conform to the mind) we also need to exploit the axioms of "Kantian transcendental apperception" (where the self and the world come together) or "Humean" one (all experience is the succession of a variety of contents):

$(\varphi \Supset \psi) \supset (\varphi \supset \psi)$

$\sim\varphi \supset \neg\varphi$

The same way as before we enrich our language with the following "inaccessibility" operators and conditions:

$a \Vdash \Xi\varphi$ iff $\forall b \ngeq^{o} a\ (b \Vdash \varphi)$ ($\Xi\varphi$ means "Nobody knows that φ" – common ignorance)

$a \Vdash \Xi_i\varphi$ iff $\exists i \in I\ \forall b \ngeq_i a\ (b \Vdash \varphi)$ (Ξ_i means "An agent *i* do not know that φ" – individual ignorance)

Dual operators would be defined with the help of definitions

$a \Vdash \mathrm{S}\varphi$ iff $\exists b \ngeq^{o} a\ (b \Vdash \varphi)$ ($\mathrm{S}\varphi$ means "Nobody believes that φ" – common disbelief)

$a \Vdash \mathrm{S}_i\varphi$ iff $\exists i \in I\ \exists b \ngeq_i a\ (b \Vdash \varphi)$ ($\mathrm{S}_i\varphi$ means "An agent *i* do not believe that φ" – individual disbelief)

Finally, we can obtain explication of reliability by way of the definitions:

$\boxplus\varphi =_{def} \Xi_i\varphi \wedge \mathrm{B}_i\varphi$ ($\boxplus\ \varphi$ means "Conceivably reliable that φ" – an agent *i* do not know that φ, and yet he believes that φ)

$\circledast\ \varphi =_{def} \mathrm{S}\varphi \vee \mathrm{K}\varphi$ ($\circledast\ \varphi$ means "Entirely reliable, that φ"- either no one convinced that φ, or everybody knows that φ)

A co-reflection $\varphi \supset \mathrm{K}\varphi$, typical for intuitionistic epistemic logic, in our Kantian version could have a form $\varphi \supset \circledast\ \varphi$ being "transcendental" consequence of axiom like $\varphi \Supset \circledast\ \varphi$. Hence, in terms of the Brouwer-Heyting-Kolmogorov (BHK) semantics it means that if φ is constructively true, i.e. has a specific proof, then φ will be *entirely reliable*.

Some metalogical results are also obtained while considering extensions of intuitionistic calculus with operators introduced.

References

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