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SIMULATIONS OF PROPOSITIONAL SINGLE-SUCCEDENT INTUITIONISTIC SEQUENT CALCULI

We consider the following single-succedent sequent calculi for intuitionistic propositional logic.

Definition 1 (ASI)**.** Sequent calculus with additive rules (cf. [1]).

Logical rules.

$$\supset_l: \frac{\Gamma \to A \quad B, \Gamma \to \Lambda}{A \supset B, \Gamma \to \Lambda}; \supset_r: \frac{\Gamma, A \to B}{\Gamma \to A \supset B};$$

$$\wedge_{l1}: \frac{A, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta}; \wedge_{l2}: \frac{B, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta}; \wedge_r: \frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B};$$

$$\vee_l: \frac{A, \Gamma \to \Theta \quad B, \Gamma \to \Theta}{A \vee B, \Gamma \to \Theta}; \vee_{r1}: \frac{\Gamma \to A}{\Gamma \to A \vee B}; \vee_{r2}: \frac{\Gamma \to B}{\Gamma \to A \vee B};$$

$$\neg_l: \frac{\Gamma \to A}{\neg A, \Gamma, \to}; \neg_r: \frac{A, \Gamma \to}{\Gamma \to \neg A}.$$

Cut rule.

$$\mathbf{cut}: \frac{\Gamma \to D \quad D, \Gamma \to \Lambda}{\Gamma \to \Lambda}.$$

Weak structural rules.

$$W_l: \frac{\Gamma \to \Theta}{D, \Gamma \to \Theta}; W_r: \frac{\Gamma \to}{\Gamma \to D}; C_l: \frac{D, D, \Gamma \to \Theta}{D, \Gamma \to \Theta}; E_l: \frac{\Delta, D, C, \Gamma \to \Theta}{\Delta, C, D, \Gamma \to \Theta}.$$

We define two versions of ASI: with proofs written as sequences of sequents (ASI_{dag}) and with proofs written as trees of sequents (ASI_{tree}).

The second calculus is a slight modification of Gentzen's formulation of LJ presented in [2].

Definition 2 (MSI)**.** Sequent calculus with multiplicative rules.

Logical rules.

$$\supset_l: \frac{\Gamma \to A \quad B, \Delta \to \Lambda}{A \supset B, \Gamma, \Delta \to \Lambda}; \supset_r: \frac{\Gamma, A \to B}{\Gamma \to A \supset B};$$

$$\wedge_{l1}: \frac{A, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta}; \wedge_{l2}: \frac{B, \Gamma \to \Theta}{A \wedge B, \Gamma \to \Theta}; \wedge_r: \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \wedge B};$$

$$\vee_l: \frac{A, \Gamma \to \Theta \quad B, \Delta \to \Theta}{A \vee B, \Gamma, \Delta \to \Theta}; \vee_{r1}: \frac{\Gamma \to A}{\Gamma \to A \vee B}; \vee_{r2}: \frac{\Gamma \to B}{\Gamma \to A \vee B};$$

$$\neg_l: \frac{\Gamma \to A}{\neg A, \Gamma, \to}; \neg_r: \frac{A, \Gamma \to}{\Gamma \to \neg A}.$$

Cut rule.

$$\mathbf{cut} : \frac{\Gamma \to D \quad D, \Delta \to \Lambda}{\Gamma, \Delta \to \Lambda}.$$

Weak structural rules.

$$W_l : \frac{\Gamma \to \Theta}{D, \Gamma \to \Theta}; W_r : \frac{\Gamma \to}{\Gamma \to D}; C_l : \frac{D, D, \Gamma \to \Theta}{D, \Gamma \to \Theta}; E_l : \frac{\Delta, D, C, \Gamma \to \Theta}{\Delta, C, D, \Gamma \to \Theta}.$$

Again, as with ASI we define MSI_{dag} and MSI_{tree}.

We prove the following propositions.

Theorem 1. *Assume, there is a* MSI_{dag}*-derivation of* $\Gamma \to \Delta$ *in* n *steps. Then there is an* ASI_{dag}*-derivation of* $\Gamma \to \Delta$ *in* $O(n^3)$ *steps.*

Theorem 2. *Assume, there is a* MSI_{dag}*-derivation of* $\Gamma \to \Delta$ *in* n *steps. Then there is an* ASI_{dag}*-derivation of* $\Gamma \to \Delta$ *in* $O(n^3)$ *steps.*

Theorem 3. *Assume, there is a* MSI_{tree}*-derivation of* $\Gamma \to \Delta$ *in* n *steps. Then there is an* ASI_{tree}*-derivation of* $\Gamma \to \Delta$ *in* $O(n^3)$ *steps.*

Theorem 4. *Assume, there is a* ASI_{dag}*-derivation of* $\Gamma \to \Delta$ *in* n *steps. Then there is an* MSI_{dag}*-derivation of* $\Gamma \to \Delta$ *in* $O(n^3)$ *steps.*

Theorem 5. *Assume, there is a* ASI_{dag}*-derivation of* $\Gamma \to \Delta$ *in* n *steps. Then there is an* ASI_{tree}*-derivation of* $\Gamma \to \Delta$ *in* $O(n^3)$ *steps.*

Theorem 6. *Assume, there is a* MSI_{dag}*-derivation of* $\Gamma \to \Delta$ *in* n *steps. Then there are such* $\Gamma' \subseteq \Gamma$ *and* $\Delta' \subseteq \Delta$ *that there is an* MSI_{tree}*-derivation of* $\Gamma' \to \Delta'$ *in* $O(n^3)$ *steps.*

References

[1] U. Egly and S. Schmitt. On intuitionistic proof transformations, their complexity, and application to constructive program synthesis. *Fundamenta Informaticae*, 39(1,2):59–83, 1999.

[2] G. Gentzen. Untersuchungen über das logische Schließen I. *Mathematische Zeitschrift*, (39):176–210, 1934.