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SIMPLIFIED KRIPKE STYLE SEMANTICS WITHOUT POSSIBLE WORLDS FOR SOME MODAL LOGICS

A history of Kripke style semantics dates back to Kripke's paper [1]. His approach introduced in [1] has some important differences from its later version presented in classical works [2, 3, 4]. First of all, in [1] Kripke dealt with **S5** only. To be more exact, with its first-order extension. Second, he did not use the accessibility relation. Thirdly, Kripke interpreted a set of all possible worlds as a set of all valuations of a formula φ. In fact, he could not postulate possible worlds and just use the notion of valuation. In [2, 3, 4] Kripke extended his approach for other systems of modal logics, using the notion of accessibility relation and another interpretation of possible worlds. The following problems were left open:

(I) Whether its possible to present Kripke style semantics *without the accessibility relation* for modal logics which differ from **S5**?

(II) Whether its possible to present Kripke style semantics such that *a set of all possible worlds is a set of all valuations of a formula* φ for modal logics which differ from **S5**?

The positive answer to question (I) was given by Pietruszczak [5]. He introduced the notion of *simplified Kripke style semantics*. The pair $\langle W, A\rangle$ is said to be a *simplified frame* iff "W is a non-empty set of worlds and $A \subseteq W$ (A is a set of *common alternatives* to all worlds from W)" [5, p. 165-166]. As follows from [5], modal logics **K45**, **KB4**, and **KD45** are determined by some classes of simplified frames.

In this report, we give the positive answer to question (II). We introduce the notions of *valuational semantics* and *valuational model*.

Let $\mathfrak{L}$ be a modal language with an alphabet $\langle\{p, q, r, p_1, \ldots\}, \neg, \supset, \Box, (,)\rangle$ such that $p, q, r, p_1, \ldots$ are atomic (propositional) variables, $\neg$ and $\supset$ are negation and implication, respectively, $\Box$ is a necessity operator. A possibility operator $\Diamond$ is defined as an abbreviation for $\neg\Box\neg$. The set of all $\mathfrak{L}$'s formulas is denoted by For. $\mathsf{At}(\varphi)$ is a set of all φ's atomic variables, where $\varphi \in \mathsf{For}$.

Following [2], we define a Kripke model as a quadruple $\langle W, a, R, \mathsf{v}\rangle$ such that W is a non-empty set of possible worlds, $a \in W$ is a distinguished possible world, $R = W \times W$ is the accessibility relation on W, and v is a valuation such that $W \times \mathsf{At}(\varphi) \longrightarrow \{0, 1\}$, where $\varphi \in \mathsf{For}$. A valuation v is extended to a valuation $V^{\mathsf{v}}\colon W \times \{\varphi\} \longrightarrow \{1, 0\}$ in a standard way.

Simplified model is a quadruple $\langle W, a, A, \mathsf{v}\rangle$ such that $\langle W, a, A\rangle$ is simplified frame and $\mathsf{v}\colon W \times \mathsf{At}(\varphi) \longrightarrow \{0, 1\}$, for each $\varphi \in \mathsf{For}$. A valuation v is extended to a valuation $V^{\mathsf{v}}\colon W \times \{\varphi\} \longrightarrow \{1, 0\}$ in the following way: V^{v} preserves all classical conditions for $\neg$ and $\supset$, moreover, for each $\varphi \in W$ and $w \in W$, $V(\Box\varphi, x) = 1$ iff $\forall_{y \in A} V(\varphi, y) = 1$.

Valuational model received from Kripke model $\mathfrak{M} = \langle W, a, R, \mathsf{v}\rangle$ for $\varphi \in \mathsf{For}$ is a triple $\langle W_{\mathfrak{M}}, a_{\mathfrak{M}}, R_{\mathfrak{M}}\rangle$ such that $W_{\mathfrak{M}} = \{\mathsf{v}_x \mid x \in W\}$, where $\mathsf{v}_x(\psi) = \mathsf{v}(x, \psi)$, for each $\psi \in \mathsf{At}(\varphi)$ and $x \in W$; $a_{\mathfrak{M}}(\psi) = \mathsf{v}(a, \psi)$, for each $\psi \in \mathsf{At}(\varphi)$; $R_{\mathfrak{M}} \subseteq W_{\mathfrak{M}} \times W_{\mathfrak{M}}$ is defined as follows, for each $\mathsf{x}, \mathsf{y} \in W_{\mathfrak{M}}$: $\mathsf{x} R_{\mathfrak{M}} \mathsf{y}$ iff $\exists_{x, y \in W}(\mathsf{x} = \mathsf{v}_x$ and $\mathsf{y} = \mathsf{v}_y$ and $xRy)$.

Note the notion of a received relation $R_{\mathfrak{M}}$ is useful only if we are able to show that the following condition is fulfiled:

$$\forall_{x, y \in W}(\mathsf{v}_x R_{\mathfrak{M}} \mathsf{v}_y \text{ iff } xRy). \tag{1}$$

We show that condition (1) is satisfied in the case of modal logics **S5**, **K45**, **KB4**, and **KD45** only. Moreover, all these logics have simplified semantics, therefore, $R = W \times A$. Thus, we prove that modal logics **K45**, **KB4**, and **KD45** are determined by some classes of valuational models such that they are special kind of simplified models which have a set of all valuations of a formula φ instead of a set of all possible worlds.

Moreover, we present simplified and valuational semantics for some regular modal logics, including **CB4**, **C45**, and **E5** (in Kripke's sence [3]).

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