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FUNCTIONAL PROPERTIES OF FOUR-VALUED PARALOGICS

Introduction. In our report we consider various features of the class of propositional literal paralogics. Literal paralogics are logics in which the paraproperties such as paraconsistence, paracompleteness and paranormality, occur only at the level of literals; that is, formulas that are propositional letters or their iterated negations.

There are several algorithms of constructing classes of literal paralogics, for example, we can emphasize the following: 1) construction of paralogics by combination of isomorphs of classical logic **CPC** [3]; 2) construction of paralogics by using literal paramatrices [5].

We turned to the first approach and regarded the class of paralogics constructed by it for the four-valued case. Then we analyzed the functional properties of the paralogics and determined the connection between functional properties of a logic (expressiveness) and its paraproperties by constructing an appropriate semi-lattice.

Paralogics obtained by combination of CPC isomorphs. DEFINITION: Some fragment of a logic **L** is said to be a **CPC** *isomorph* iff **L** has the classical set of tautologies and the classical consequence relation.

It is known that Bochvar's three-valued nonsense logic $\mathbf{B}_3$ includes two **CPC** isomorphs and the combination of these two isomorphs leads to construction of two famous paralogics – $\mathbf{P}^1$ and $\mathbf{I}^1$, which are *functionally equivalent*. Moreover, each of these logics is functionally equivalent to the fragment of logic $\mathbf{B}_3$ consisting of external formulas only (for a more detailed account see [4]).

Let us now consider four-valued Bochvar's logic $\mathbf{B}_4$ determined by the matrix

$\mathfrak{M}_4^B = \langle \{0, 1/3, 2/3, 1\}, \sim, \cap, \cup, J_0, J_{1/3}, J_{2/3}, J_1, \{1\} \rangle$,

where $\sim x = 1 - x$, and J-operators, $\cap$ and $\cup$ are defined by the following truth-tables (cf. [1, p. 294]):

x	$J_0(x)$	$J_{1/3}(x)$	$J_{2/3}(x)$	$J_1(x)$
1	0	0	0	1
2/3	0	0	1	0
1/3	0	1	0	0
0	1	0	0	0

$\cap$	1	2/3	1/3	0
1	1	2/3	1/3	0
2/3	2/3	2/3	1/3	1/3
1/3	1/3	1/3	1/3	1/3
0	0	1/3	1/3	0

$\cup$	1	2/3	1/3	0
1	1	2/3	2/3	1
2/3	2/3	2/3	2/3	2/3
1/3	2/3	2/3	1/3	1/3
0	1	2/3	1/3	0

Functional properties of Bochvar's logic $\mathbf{B}_3$ are determined by the union of two types of connectives – internal and external (their truth-tables contain values 0 and 1 only). In the three-valued case internal connectives can be translated into external ones in two different ways [4, p. 7]. These two translations provide construction of two fragments of $\mathbf{B}_3$ isomorphic with **CPC**. In the one isomorph the truth-value 1/2 is identified with 0 and in the other – with 1.

In the four-valued case there are four translation functions: $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$:

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
1	1	1	1	1
2/3	0	1	1	0
1/3	0	1	0	1
0	0	0	0	0

Using $f_1(x)$, $f_2(x)$, $f_3(x)$ and $f_4(x)$ analogously as it was done for $\mathbf{B}_3$, we can construct four external negations and four external implications:

$\neg_i x := \sim f_i(x)$ and $x \to_i y := \neg_i x \cup f_i(y)$, where $i \in \{1, 2, 3, 4\}$.

Four-valued **CPC** isomorphs are determined by the following matrices:

$\mathfrak{M}_1 = \langle\{0, 1/3, 2/3, 1\}, \neg_1, \to_1, \{1\}\rangle$, $\mathfrak{M}_2 = \langle\{0, 1/3, 2/3, 1\}, \neg_2, \to_2, \{1, 2/3, 1/3\}\rangle$,
$\mathfrak{M}_3 = \langle\{0, 1/3, 2/3, 1\}, \neg_3, \to_3, \{1, 2/3\}\rangle$, $\mathfrak{M}_4 = \langle\{0, 1/3, 2/3, 1\}, \neg_4, \to_4, \{1, 1/3\}\rangle$.

Combining the operations $\neg_i$, $\to_j$ of the isomorphs we can construct the class of four-valued literal paralogics.

paraconsistent logics
$\mathfrak{M}_5 = \langle\{0, 1/3, 2/3, 1\}, \neg_1, \to_2, \{1, 2/3, 1/3\}\rangle$,
$\mathfrak{M}_6 = \langle\{0, 1/3, 2/3, 1\}, \neg_3, \to_2, \{1, 2/3, 1/3\}\rangle$,
$\mathfrak{M}_7 = \langle\{0, 1/3, 2/3, 1\}, \neg_4, \to_2, \{1, 2/3, 1/3\}\rangle$,
$\mathfrak{M}_8 = \langle\{0, 1/3, 2/3, 1\}, \neg_1, \to_3, \{1, 2/3\}\rangle$,
$\mathfrak{M}_9 = \langle\{0, 1/3, 2/3, 1\}, \neg_1, \to_4, \{1, 1/3\}\rangle$.

paracomplete logics
$\mathfrak{M}_{10} = \langle\{0, 1/3, 2/3, 1\}, \neg_2, \to_1, \{1\}\rangle$,
$\mathfrak{M}_{11} = \langle\{0, 1/3, 2/3, 1\}, \neg_3, \to_1, \{1\}\rangle$,
$\mathfrak{M}_{12} = \langle\{0, 1/3, 2/3, 1\}, \neg_4, \to_1, \{1\}\rangle$,
$\mathfrak{M}_{13} = \langle\{0, 1/3, 2/3, 1\}, \neg_2, \to_3, \{1, 2/3\}\rangle$,
$\mathfrak{M}_{14} = \langle\{0, 1/3, 2/3, 1\}, \neg_2, \to_4, \{1, 1/3\}\rangle$.

paranormal logics
$\mathfrak{M}_{15} = \langle\{0, 1/3, 2/3, 1\}, \neg_4, \to_3, \{1, 2/3\}\rangle$,
$\mathfrak{M}_{16} = \langle\{0, 1/3, 2/3, 1\}, \neg_3, \to_4, \{1, 1/3\}\rangle$.

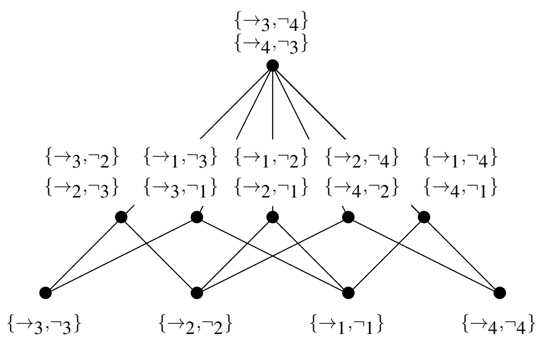
And logics $\mathfrak{M}_1$–$\mathfrak{M}_4$ are neither paraconsistent nor paracomplete. They are included in our class of paralogics as a degenerate case.

The study of the functional properties of the presented class of paralogics and finding their connection with paraproperties is of special interest.

Method to find the functional inclusion. Logic $\mathbf{L}_x$ is functionally included in logic $\mathbf{L}_y$ iff the basis of $\mathbf{L}_x$ is included in the set of functions of $\mathbf{L}_y$. That is, to answer the question about the functional inclusion it is enough to determine whether the two basis functions of $\mathbf{L}_x$ are contained in the set of $\mathbf{L}_y$ functions.

We solved that task as follows. First, we constructed a program model that, given two sequences of operations of a logic, generates the result of their application to the two fixed arguments. This modeling method was already used to verify cache coherence protocols in [6] and to verify cryptographic protocols in [7]. Then, using a supercompiler [8], we unfolded the partial computation tree [2, p. 113] of the program model, listing the functions generated along the paths of the tree.

Lattice of four-valued paralogics. The results of our analysis show that the sixteen paralogics listed in Section form the upper semi-lattice with respect to the functional inclusion.



Semi-lattice of four-valued paralogics

From the functional point of view the class of paralogics considered is divided into 10 equivalence classes and form 10-element upper semi-lattice. There are five different equivalence classes, every class includes two equivalent paralogics – paraconsistent and paracomplete. The equivalence class that includes two paranormal logics is the supremum of the semi-lattice and has the greatest expressive power in the whole class of paralogics obtained by the combination of isomorphs. And this class is functionally equivalent to the set of all four-valued external functions.

References

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