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NATURAL DEDUCTION RULES FOR TOMOVA'S NATURAL IMPLICATIONS

The class of Tomova's natural logics is described in [11, 10, 3]. These logics are built over a propositional language $\mathfrak{L}$ with the following alphabet: $\langle\{p_1, p_2, \ldots\}, \neg, \rightarrow, \wedge, \vee, (,)\rangle$. The notion of a $\mathfrak{L}$-formula is defined in a standard way. The $\{\neg, \wedge, \vee\}$-fragments of Tomova's natural logics are three-valued regular Kleene's logics. Implications of Tomova's logics are natural in the following sense [3, p. 210-211].

Definition 1 (Natural implication)**.** Let V_3 be the set $\{1, 1/2, 0\}$ of truth values and D be the set of designated values such that either $D = \{1\}$ or $D = \{1, 1/2\}$. Then implication $\rightarrow$ is called *natural* iff the following conditions hold:

(1) *C-extending*, i.e. its restrictions to the subset $\{0, 1\}$ of V_3 coincide with classical implication;

(2) *normality* in the sense of Łukasiewicz-Tarski, i.e. for all $x, y \in V_3$: if $x \rightarrow y \in D$ and $x \in D$, then $y \in D$ (the condition that is sufficient for the verification of modus ponens) [6, p. 134];

(3) *consistency*, i.e. for all $x, y \in V_3$: if $x \leqslant y$, then $x \rightarrow y \in D$.

As follows from the definition of natural implication, there are 6 implications with $D = \{1\}$ and 24 implications with $D = \{1, 1/2\}$.

In this report, we consider two three-valued regular (in Kleene's sence [4]) logics: strong Kleene's logic $\mathbf{K_3}$ [4] and Asenjo & Priest's logic of paradox **LP** [1, 8]. However, note the class of all three-valued regular logics is bigger (see [5]) and Tomova deals with all of its elements. $\mathbf{K_3}$ and **LP** are built in $\mathfrak{L}$'s $\{\neg, \wedge, \vee\}$-fragment and their connectives are defined as follows: $v(\neg\alpha) = 1 - v(\alpha)$, $v(\alpha \wedge \beta) = min(v(\alpha), v(\beta))$, and $v(\alpha \vee \beta) = max(v(\alpha), v(\beta))$, where α, β are formulas and v is a valuation. In the case of $\mathbf{K_3}$, $D = \{1\}$ while in the case of **LP**, $D = \{1, 1/2\}$. In $\mathbf{L} \in \{\mathbf{K_3}, \mathbf{LP}\}$, the entailment relation is defined as follows: $\Gamma \models_{\mathbf{L}} \alpha$ iff for each valuation v, it holds that if $v(\gamma) \in D$ (for each $\gamma \in \Gamma$), then $v(\alpha) \in D$, where Γ is a set of formulas and α is formula.

Natural deduction systems for $\mathbf{K_3}$ and **LP** were presented by Priest [8]. Let $\mathfrak{R}$ be a set of the following inference rules:

$$(EM)\,\frac{}{\alpha \vee \neg\alpha} \quad (EFQ)\,\frac{\alpha \wedge \neg\alpha}{\beta} \quad (\neg\neg I)\,\frac{\alpha}{\neg\neg\alpha} \quad (\neg\neg E)\,\frac{\neg\neg\alpha}{\alpha}$$

$$(\vee I_\alpha)\,\frac{\alpha}{\alpha \vee \beta} \quad (\vee I_\beta)\,\frac{\beta}{\alpha \vee \beta} \quad (\vee E)\,\frac{\alpha \vee \beta \quad \begin{matrix}[\alpha]\\ \gamma\end{matrix} \quad \begin{matrix}[\beta]\\ \gamma\end{matrix}}{\gamma}$$

$$(\wedge I)\,\frac{\alpha \quad \beta}{\alpha \wedge \beta} \quad (\wedge E_\alpha)\,\frac{\alpha \wedge \beta}{\alpha} \quad (\wedge E_\beta)\,\frac{\alpha \wedge \beta}{\beta}$$

$$(\neg\vee I)\,\frac{\neg\alpha \wedge \neg\beta}{\neg(\alpha \vee \beta)} \quad (\neg\vee E)\,\frac{\neg(\alpha \vee \beta)}{\neg\alpha \wedge \neg\beta} \quad (\neg\wedge I)\,\frac{\neg\alpha \vee \neg\beta}{\neg(\alpha \wedge \beta)} \quad (\neg\wedge E)\,\frac{\neg(\alpha \wedge \beta)}{\neg\alpha \vee \neg\beta}$$

As follows from [8], $\mathfrak{R} \setminus \{(EM)\}$ and $\mathfrak{R} \setminus \{(EFQ)\}$, respectively, are sets of inference rules for $\mathbf{K_3}$ and **LP**. The notion of an inference is defined in a linear format standardly (see [2] for a textbook-style definition and [9] for a precise one).

Let $\mathbf{K_3}_{\to}$ and $\mathbf{LP}_{\to}$, respectively, be $\mathbf{K_3}$'s and $\mathbf{LP}$'s extensions by an implication $\to$. Logic is said to be *natural* iff it is $\mathbf{K_3}$'s or $\mathbf{LP}$'s extension by *natural* implication. Thus, $\mathbf{K_3}_{\to}$ and $\mathbf{LP}_{\to}$ are natural logics iff $\to$ is natural implication. Besides, let $\mathfrak{ND}_{K_{3\to}}$ and $\mathfrak{ND}_{LP_{\to}}$ be natural deduction systems for $\mathbf{K_3}_{\to}$ and $\mathbf{LP}_{\to}$, respectively. In this report, we will present all the inference rules for $\mathfrak{ND}_{K_{3\to}}$ and $\mathfrak{ND}_{LP_{\to}}$. Now consider the following ones:

$$(R_{1/1})\ \frac{(\alpha \vee \neg\alpha) \wedge (\beta \vee \neg\beta)}{(\alpha \to \beta) \vee \neg(\alpha \to \beta)} \qquad (R_{1/2})\ \frac{(\alpha \to \beta) \wedge \neg(\alpha \to \beta)}{(\alpha \wedge \neg\alpha) \vee (\beta \wedge \neg\beta)}$$

$$(MP)\ \frac{\alpha \quad \alpha \to \beta}{\beta} \qquad (R_3)\ \frac{\begin{array}{cc}[\alpha] & [\neg\beta] \\ \beta & \neg\alpha\end{array}}{\alpha \to \beta}$$

Our main result is presented below.

Theorem 2. *A logic* $\mathbf{K_3}_{\to}$ *is natural iff the rules* $(R_{1/1})$, (MP), *and* (R_3) *are derivable in* $\mathfrak{ND}_{K_{3\to}}$. *A logic* $\mathbf{LP}_{\to}$ *is natural iff the rules* $(R_{1/2})$, (MP), *and* (R_3) *are derivable in* $\mathfrak{ND}_{LP_{\to}}$.

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