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YABLO'S PARADOX, THE LIAR, AND REFERENTIAL CONTRADICTIONS FROM A GRAPH THEORY POINT OF VIEW

Abstract. F-systems are useful digraphs to model sentences that predicate the falsity of other sentences. Paradoxes like the Liar and the one of Yablo can be analyzed with that tool to find graph-theoretic patterns. In this paper we studied this general model consisting of a set of sentences and the binary relation '... affirms the falsity of...' among them. The possible existence of non-referential sentences was also considered. To model the sets of all the sentences that can jointly be valued as true we introduced the notion of *conglomerate*, the existence of which guarantees the absence of paradox. Conglomerates also enabled us to characterize *referential contradictions*, i.e., sentences that can only be false under a classical valuation due to the interactions with other sentences in the model. A Kripke-style fixed-point characterization of groundedness was offered, and complete (meaning that every sentence is deemed either true or false) and consistent (meaning that no sentence is deemed true and false) fixed points were put in correspondence with conglomerates. Furthermore, argumentation frameworks are special cases of F-systems. We showed the relation between *local conglomerates* and admissible sets of arguments and argued about the usefulness of the concept for the argumentation theory.
Keywords: the Liar paradox, Yablo's paradox, F-system, conglomerates, groundedness, argumentation frameworks.

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Rabern, Rabern, Macauley (2013), inspired by Cook (2004), coined the term "F-systems" to refer to "sentence systems which are restricted in such a way that all the sentences can only say that other sentences in the system are false". So, F-systems can be represented as digraphs that model the interaction among sentences that predicate falsity of other sentences (call them *falsity predicate sentences*). An *F-system* is a pair $F = \langle S, F \rangle$, where S is a set of primitive entities called *sentences*, and F is a binary relation on S intended to mean '...affirms the falsity of...'. This is the only sense in which sentences can refer to other sentences in the model. The existence of non-falsity predicate sentences is also considered (cf. Beringer, Schindler 2017). These can be referred to by other sentences, but they do not refer to other sentences (i.e. they are *sinks* in the digraph).

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Paradoxes like the Liar and the one of Yablo can be analyzed with that tool to find graph-theoretic patterns. Paradoxes can be characterized by means of assignment of truth values in a way such that a sentence is assigned 'false' if it refers to some sentence to which is assigned 'true', and it is assigned 'true' if all the sentences it refers to are assigned 'false'. If a sentence cannot be assigned either 'true' or 'false', then it is paradoxical.

I introduce the notion of *conglomerate* to model maximal (w.r.t. $\subseteq$) subsets of sentences that can jointly be valued as true. Briefly, a conglomerate is a set of sentences such that none of its sentences refer to other sentences within the set, and every outer non-sink sentence refers to some inner sentence. More formally, let $\overleftarrow{F}(A) = \{x : (z, x) \in F \text{ for some } z \in A\}$. Then:

Definition 1. Given $F = \langle S, F \rangle$, a *conglomerate* is a subset $A \subseteq S$ that satisfies:

1. $\overleftarrow{F}(A) \subseteq S \quad A$ (*independence*), and
2. $(S \backslash A) \backslash sinks(S) \subseteq \overleftarrow{F}(A)$ (*non-sinks absorption*)

The existence of conglomerates in a system guarantees the absence of paradox— at least, those generated by the falsity predicate. The Liar paradox, for instance, can be represented as an F-system $\langle \{a\}, \{(a, a)\} \rangle$ (i.e., a expresses its own falsity). Yablo's paradox, in turn, can be represented as $\langle \{a_k\}_{k \in \mathbb{N}}, \{(a_k, a_m)\}_{k<m} \rangle$ (i.e. an infinite sequence of sentences, each one referring to the falsity of each later sentence). There are no conglomerates in any of those settings.

The notion of conglomerate extends that of *kernel* used by Cook (2004). Kernels are such that *all* external sentences refer to some internal sentence. Kernels are suitable for capturing classical truth values assignments in systems where *each* sentence refers to some other sentence. But kernels do not fit well in systems that include sinks. The reason is that sinks always belong to every kernel and, in consequence, kernels can only model them as true. But sinks are intended to represent sentences of varied truth values, like 'Snow is white' or 'It rains in Moscow right now'. For example, in a system with only two sentences, 'It rains in Moscow right now' and 'It is false that it rains in Moscow right now' (i.e., the latter affirms the falsity of the first one), we will have two conglomerates, each containing only one of the sentences, representing the fact that either one of them is true and the other one false, or vice versa; however, we will only have one kernel containing 'It rains in Moscow right now', which only enables to interpret that sentence as true, and the other one as false.

Since paradoxical F-systems have no conglomerates, this notion is not well-defined. However, we can still want to know what sentences can be true together in systems containing paradoxes, even if that class is empty. Possibly true sentences can be captured by defining a *local* version of the non-sinks absorption condition. Let $\overrightarrow{F}(A) = \{x : (x, z) \in F \text{ for some } z \in A\}$. Then:

Definition 2. Given $F = \langle S, F \rangle$, a *local conglomerate* is a subset $A \subseteq S$ that satisfies:

1. $\overleftarrow{F}(A) \subseteq S \quad A$, and

2. $\overrightarrow{F}(A) \backslash sinks(S) \subseteq \overleftarrow{F}(A)$.

Since the empty set is always a local conglomerate, the notion is well defined. Paradoxical sentences cannot belong to any local conglomerate, nor can they refer to sentences that belong to any local conglomerate (otherwise, they could be consistently assigned the false truth value and, hence, they would be not paradoxical).

Besides paradoxes, local conglomerates also allow to characterize *referential contradictions* and *referential tautologies*. Referential contradictions (resp. tautologies) are sentences that can only take the false (resp. true) value under classical valuations, due to the interactions with other sentences in the model. For example, assume p refers to both q and r, and q refers to r. Then p is a referential contradiction, since it can only have the false value in every classical (i.e. binary) valuation. For their part, referential tautologies only exist by reference to referential contradictions (for example, a sentence s referring to p in the above scenario). As a result, referential tautologies belong to every maximal (w.r.t. $\subseteq$) local conglomerate while referential contradictions are excluded from them. But referential contradictions, unlike paradoxical sentences, are "absorbed" by every maximal local conglomerate.

Transitivity is a source of pathologies in F-systems. It is a sufficient condition for referential contradictions in non-paradoxical scenarios, while it is sufficient for paradox in sink-free systems—as shown by Cook. Another source of problems is odd-length cycles, which lead to paradox whenever the sentences in the cycle are not "absorbed" by local conglomerates. While Yablo's paradox suffers from the transitivity related pathology, the Liar is doubly pathological, since it suffers both the transitivity and the odd-length cycle conditions.

Finally, maximal local conglomerates can be put in correspondence with maximal consistent fixed points of a Kripke-style "jump" operator. A *partial set* is a pair (S^+, S^-), where S^+ is intended to contain only true sentences while S^- is intended to contain only false sentences. I define an operator ϕ which applied to a partial set (S^+, S^-) returns another partial set (S'^+, S'^-), where S'^+ collects all the sinks in S^+ together with the sentences whose references are all contained in S^-, and S'^- collects all the sinks in S^- together with the sentences whose references are some contained in S^+. When ϕ returns the same partial set, we have a fixed point. Fixed points are *consistent* if no sentence is in both sets of the pair.

References

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