ЛОГИКА СЕГОДНЯ

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A. N. PRIOR'S SYSTEM Q: A REVIEW

Abstract. In his "Time and Modality", based on his own philosophical motivations, Arthur Norman Prior proposed the modal logic Q as a correct modal logic in 1957. Prior developed Q in order to offer a logic for contingent beings, in which one could rationally state that some beings are contingent and some are necessary. One may say that Q is an actualist modal logic with a natural semantics. This review article is a developed description/discussion of/on "The System Q" that is the fifth chapter of "Time and Modality". I have attempted to analyse the logical structure of system Q in order to provide a more understandable description as well as logical analysis for today's logicians, philosophers, and information-computer scientists. In the paper, the Polish notations are translated into modern notations in order to be more comprehensible and to support the developed formal descriptions and semantic analysis.
Keywords: Logical System Q, Modal Logic, Tense Logic, Three-Valued Semantics.

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Arthur Norman Prior was born on 4 December 1914 in Masterton, New Zealand. He studied philosophy in the 1930s and was a significant, and often provocative, voice in theological debates until well into the 1950s. He became a lecturer in philosophy at Canterbury University College in Christchurch in 1946 succeeding Karl Popper. He became a full professor in 1952. He left New Zealand permanently for England in 1959, first taking a chair in philosophy at Manchester University, and then becoming a fellow of Balliol College, Oxford, in 1966. Prior died on 6 October 1969 in Trondheim, Norway. After Prior's death, many logicians and philosophers have analysed and discussed his approach to formal and philosophical logic. In particular, his contributions to modal logic, tense-logic and deontic logic have been studied.

In 1957, A. N. Prior proposed the three-valued modal logic Q as a 'correct' modal logic from his philosophical motivations, see Prior (1957). Prior developed Q in order to offer a logic for contingent beings, in which one could intelligibly and rationally state

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that some beings are contingent and some are necessary, see Akama, Nagata (2005). According to Akama, Nagata (2005), Q has a natural semantics. In other words, from the philosophical point of view, Q can be regarded as an 'actualist' modal logic.

This review article is a developed description of, and discussion on, '*The System* Q' that is the fifth chapter of Prior (1957). In addition, in his logical analysis of '*Time & Existence*' (that is the eights chapter of Prior (1967)), Prior has worked on system Q. Thus, Prior (1967) has also been very useful for this article. This article analyses the logical structure of system Q in order to provide a more understandable description as well as logical analysis for today's logicians, philosophers, and information-computer scientists. In the paper, the Polish notations are translated into modern notations in order to be more comprehensible and to support the developed formal descriptions as well as semantic analysis.

I. The System Q

In Prior (1957), Prior begins by supposing that there are only two times; today and yesterday. He assumes that today is the day on which we are using our formal system. Hence today's variables can stand for any statement which we can make today (whether we could have made it yesterday or not). Note that the possibility that some of our statements could not have been made/expressed yesterday enlarges the possible values of our statements. Accordingly, a statement, say p, can totally have six possible values:

i. The statement p is true at both times. In our logical system the value 'True on yesterday and True on today' can be represented by $\mathrm{T_y} - \mathrm{T_t}$.
ii. The statement p is true today and was unstatable, i.e. vague, yesterday. The value of p can be represented by $\mathrm{V_y} - \mathrm{T_t}$.
iii. The statement p is true today and was false yesterday. Then the value is $\mathrm{F_y} - \mathrm{T_t}$.
iv. The statement p is false today but was true yesterday. Then the value is $\mathrm{T_y} - \mathrm{F_t}$.
v. The statement p is false today and was unstatable (vague) yesterday. So the value is $\mathrm{V_y} - \mathrm{F_t}$.
vi. The statement p has been false at both times. Therefore, the value is $\mathrm{F_y} - \mathrm{F_t}$.

In addition, if p is a statement which could not have been made/expressed yesterday, then any compound statement containing it, such as '$\sim p$' or '$p \wedge q$' or '$\Diamond p$' or '$\Box p$', could not have been made yesterday either. In our formalism $\Diamond p$ is utilised to express 'p at some time'. In addition, $\Box p$ expresses 'p [at] both times'.

Definition. The sentences (1) 'p is true today and was true yesterday' and (2) 'p is true today and was unstatable/vague yesterday are *designated*. The reason is that a formula expresses a logical law if its concrete substitutions are true whenever they are statable.

According to the afore-mentioned principles, it is easy to construct semantic tables for conjunction ($\wedge$), negation ($\sim$), possibility ($\Diamond$), and necessity ($\Box$), and work out other for implication ($\supset$), disjunction ($\vee$), equivalency ($\leftrightarrow$), etc. Now we can represent the tables for negation ($\sim$), possibility ($\Diamond$), necessity ($\Box$) and 'the negation of possibly not' ($\sim\Diamond\sim$) as follows:

p	$\sim p$	$\Diamond p$	$\Box p$	$\sim\Diamond\sim p$
value 1: *$T_y - T_t$	$F_y - F_t$	$T_y - T_t$	$T_y - T_t$	$T_y - T_t$
value 2: *$V_y - T_t$	$V_y - F_t$	$V_y - T_t$	$V_y - F_t$	$V_y - T_t$
value 3: $F_y - T_t$	$T_y - F_t$	$T_y - T_t$	$F_y - F_t$	$F_y - F_t$
value 4: $T_y - F_t$	$F_y - T_t$	$T_y - T_t$	$F_y - F_t$	$F_y - F_t$
value 5: $V_y - F_t$	$V_y - T_t$	$V_y - F_t$	$V_y - F_t$	$V_y - F_t$
value 6: $F_y - F_t$	$T_y - T_t$	$F_y - F_t$	$F_y - F_t$	$F_y - F_t$

Table 1: The Q System's Six-Valued Table

Note. In the main text of "The System Q" (Prior 1957) Prior has associated the truth, falsity, and unstatability of a proposition at the times 'today and yesterday' (more particularly, the values '$T_y - T_t$', '$V_y - T_t$', '$F_y - T_t$', '$T_y - F_t$', '$V_y - F_t$', and '$F_y - F_t$') with the natural numbers 1, 2, 3, 4, 5, 6, respectively.

It is remarkable that the only difference between the tables for $\Box p$ (column 4 of Table 1) and $\sim\Diamond\sim p$ (column 5 of Table 1) is where p has the value '$V_y - T_t$' (the second row of Table 1). In fact, in this case p is not true [at] *both* times, and today we can state that it is not (in fact, can deny) $\Box p$, but yesterday it was unstatable/vague, so that $\Box p$ has the value '$V_y - F_t$'. But in the same case, where p has the value '$V_y - T_t$', not-p has the value '$V_y - F_t$'; so that not-p is not true at either time, and this fact—that not-p is not true at either time—is true today but was unstatable/vague yesterday, i.e. $\sim\Diamond\sim p$ has the value '$V_y - T_t$'. From this difference others follow.

Definition. The symbol α expresses a *law* if and only if it always has one of the values '$T_y - T_t$' and '$V_y - T_t$'. Note that a law cannot have the value '$F_y - T_t$' (although today's value is true).

Considering α a law, it is concludable that $\sim\Diamond\sim p$ always expresses a law too (i.e. has one of the values '$T_y - T_t$' and '$V_y - T_t$'), But the same is not valid in the case of $\Box\alpha$. More specifically, $\Box\alpha$ never takes the value '$V_y - T_t$' at all. Actually $\Box\alpha$ could *only* have the value '$T_y - T_t$' (for all values of its variables) if α itself had the value '$T_y - T_t$' (for all values of its variables). This in turn could only be so if the variables in α were not capable of taking either of the values '$V_y - T_t$' and '$V_y - F_t$'; for any formula of which any part takes one of these values must itself take one of them. Note that this follows from the fact that any function of what was vague (and unstatable) yesterday will itself have been vague (and unstatable) yesterday. But it shall be taken into consideration that ordinary propositional variables are not restricted in their possible values to those other than '$V_y - T_t$' and '$V_y - F_t$'.

Proposition. In the system Q (as a system in '∼', '∧', '□' and '◊' with ordinary propositional variables only) there can be no laws of the form $\Box\alpha$ at all.

In this it is like the Ł-modal system (see Łukasiewicz 1953; Smiley 1961); but it is unlike it in that in the Ł-modal system there are not only no laws of the form $\Box\alpha$ but no individual true proposition of the form $\Box p$ (the values of such propositions being confined, in that system, to '$V_y - T_t$', '$F_y - T_t$', and '$T_y - F_t$'). In the system Q, on the other hand, $\Box p$ does have the value '$T_y - T_t$' when p has it, and there are statements for which p can stand which do have this value, though there are no forms which have it for all values, including '$V_y - T_t$' an '$V_y - F_t$', of their constituent variables.

The tables for conjunction (∧), implication (⊃), and disjunction (∨) are as follows:

∧	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$	$T_y - F_t$	$V_y - F_t$	$F_y - F_t$
*$T_y - T_t$	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$	$T_y - F_t$	$V_y - F_t$	$F_y - F_t$
*$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - F_t$	$V_y - F_t$	$V_y - F_t$
$F_y - T_t$	$F_y - T_t$	$V_y - T_t$	$F_y - T_t$	$F_y - F_t$	$V_y - F_t$	$F_y - F_t$
$T_y - F_t$	$T_y - F_t$	$V_y - F_t$	$F_y - F_t$	$T_y - F_t$	$V_y - F_t$	$F_y - F_t$
$V_y - F_t$	$V_y - F_t$	$V_y - F_t$	$V_y - F_t$	$V_y - F_t$	$V_y - F_t$	$V_y - F_t$
$F_y - F_t$	$F_y - F_t$	$V_y - F_t$	$F_y - F_t$	$F_y - F_t$	$V_y - F_t$	$F_y - F_t$

Table 2: The Conjunction Table for System Q

⊃	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$	$T_y - F_t$	$V_y - F_t$	$F_y - F_t$
*$T_y - T_t$	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$	$T_y - F_t$	$V_y - F_t$	$F_y - F_t$
*$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - F_t$	$V_y - F_t$	$V_y - F_t$
$F_y - T_t$	$T_y - T_t$	$V_y - T_t$	$T_y - T_t$	$T_y - F_t$	$V_y - F_t$	$T_y - F_t$
$T_y - F_t$	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$
$V_y - F_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$
$F_y - F_t$	$T_y - T_t$	$V_y - T_t$	$T_y - T_t$	$T_y - T_t$	$V_y - T_t$	$T_y - T_t$

Table 3: The Implication Table for System Q

∨	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$	$T_y - F_t$	$V_y - F_t$	$F_y - F_t$
*$T_y - T_t$	$T_y - T_t$	$V_y - T_t$	$T_y - T_t$	$T_y - T_t$	$V_y - T_t$	$T_y - T_t$
*$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$
$F_y - T_t$	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$
$T_y - F_t$	$T_y - T_t$	$V_y - T_t$	$T_y - T_t$	$T_y - F_t$	$V_y - F_t$	$T_y - F_t$
$V_y - F_t$	$V_y - T_t$	$V_y - T_t$	$V_y - T_t$	$V_y - F_t$	$V_y - F_t$	$V_y - F_t$
$F_y - F_t$	$T_y - T_t$	$V_y - T_t$	$F_y - T_t$	$T_y - F_t$	$V_y - F_t$	$F_y - F_t$

Table 4: The Disjunction Table for System Q

Let us focus on the implication $\sim\Diamond\sim p \supset p$. Suppose that p has the value '$V_y - T_t$'. According to Table 1 (and as discussed), the values of $\sim\Diamond\sim p \supset p$ and $\Box p$ are '$V_y - T_t$' and '$V_y - F_t$', respectively. Therefore, we can semantically conclude that the value '$V_y - T_t$' implies '$V_y - F_t$'. Regarding Table 3, the value of the implication $(V_y - T_t) \supset (V_y - F_t)$ is '$V_y - F_t$'. Hence the implication is now found to take the value '$V_y - F_t$' (which is not a designated value). It is interesting that the implication '$\sim\Box\sim p \supset \Diamond p$' takes the same value, i.e. '$V_y - F_t$', when p has the value '$V_y - F_t$'. Note that these results correspond to the intuitive objections to the S5 type of tense-logic. The system S5 is one of five systems of modal logic and is formed with propositional calculus formulae and tautologies as well as inference apparatus with substitution and modus ponens. However, its syntax has the modal operators $\Box$ and $\Diamond$, see Lemmon (1956); Chellas (1980); Hughes, Cresswell (1968). In next section we will be more specific.

Now suppose that p is something like 'I am both a logician and not a logician'. This proposition is false whenever it is statable but is not always statable. Also, $\sim p$ is not statable when p is not (and so is not always true), so that 'Not always not p' (formally speaking, $\sim\Box\sim p$) is true when statable (i.e. has the value '$V_y - T_t$'); while since p is never true, 'Sometimes p' (formally speaking, $\Diamond p$) is false when it is statable, i.e. the value '$V_y - F_t$'. And, based on Table 3, the value of the implication '$V_y - T_t \supset V_y - F_t$' is '$V_y - F_t$'. Therefore it shall be concluded that the implication $\sim\Box p \supset \Box\sim\Box p$ has the undesignated value '$V_y - F_t$' when p has either the value '$V_y - T_t$' or '$V_y - F_t$' (in both cases yesterday's value is vague/unstatable).

It is worth mentioning that the afore-mentioned implications are Gödelian axioms (and that's why we have mainly focused on them). Moreover, the other two Gödelian axioms, and the classical propositional calculus, are verified; but besides being more tedious to work out, this result is of less significance, for although whatever is falsified by these tables will be something we do not want in Q, we will not want all that they verify, since some formulae which they verify merely reflect the fiction that there are 'only' two times (and 'we really do not want to say that we only have two times').

II. The Matrix M_Q

Proposition 1. The analysed 6-valued tables (based on the values (i) $T_y - T_t$, (ii) $V_y - T_t$, (iii) $F_y - T_t$, (iv) $T_y - F_t$, (v) $V_y - F_t$, (vi) $F_y - F_t$) are, in fact, just the first step towards the matrix with an 'infinite number of elements' which would give us the exact 'many-valued' equivalent of the system Q. This matrix, let us call it 'M_Q', may be described as follows:

(1) Each of M_Q's elements may be associated with an infinite sequence of the symbols 'T' (that stands for True), 'V' (that stands for Vague), and 'F' (that stands for False) and of which the last symbol (that expresses the value of today's) must be T or F (because there is no unstatability/vagueness for today's proposition).

Intuitively we can think of these symbols as indicating whether a proposition is true, unstatable, or false at each one of an infinity of times.

(2) The sequence for $\sim p$ is determined by the sequence for p as follows:

$$\begin{aligned}
&\sim(x) = (\sim x),\\
&\sim(xy) = (\sim x)(\sim y),\\
&\sim(xyz) = (\sim x)(\sim y)(\sim y),\\
&\ldots,\\
&\sim(xyz\ldots) = (\sim x)(\sim y)(\sim z)\ldots,
\end{aligned}$$

where the Table for 'negation' considered as operating on a single symbol is:

x	$\sim x$
T	F
V	F
F	T

(3) The sequence for $p \wedge q$ is determined by the sequences for p and for q as follows:

$$\begin{aligned}
&(x) \wedge (x') = (x \wedge x'),\\
&(xy) \wedge (x'y') = (x \wedge x')(y \wedge y'),\\
&(xyz) \wedge (x'y'z') = (x \wedge x')(y \wedge y')(z \wedge z'),\\
&\ldots,\\
&(xyz\ldots) \wedge (x'y'z'\ldots) = (x \wedge x')(y \wedge y')(z \wedge z')\ldots,
\end{aligned}$$

where Table 2, considered as operating on a pair of single value, is:

$\wedge$	T	V	F
T	T	V	F
V	V	V	V
F	F	V	F

(4) The sequence for $\Box p$ is determined by the sequence for p as follows:

(a) If the sequence for p is that consisting of T's only the sequence for $\Box p$ is surely the same.

(b) If the sequence for p contains any V's, then in the sequence for $\Box p$ these V's keep their place unaltered, and all other places are occupied by F's (in fact, it does not matter if they have been 'F' or 'T' in the sequence for p. In the sequence for $\Box p$, both will be occupied by 'F').

  (c) If the sequence for p contains no V's, but contains F's, whether it also contains T's or not, the sequence for $\Box p$ is that consisting of F's only.

(5) The sequence for $\Diamond p$ is determined by the sequence for p as follows:

  (a) Where the sequence for p consists either of F's only or of F's and V's only, the sequence for $\Diamond p$ is the same as the sequence for p.
  (b) Where the sequence for p contains V's and T's, whether it contains F's or not, in the sequence for $\Diamond p$ the V's keep their place unaltered and all other places are occupied by T's.
  (c) Where the sequence for p contains no V's, but does contain T's, whether it contains F's or not, the sequence for $\Diamond p$ is that consisting of T's only.

(6) The designated sequences are all those which contain no F's. In fact, there should not be any falsity in a designated sequence.

The way in which these rather complicated stipulations preserve the characteristic features of our 6-valued system Q, apart from those which reflect its underlying fiction of a limited number of times, may be illustrated by 'the non-equivalence of $\Box p$ and $\sim\Diamond\sim p$'. The important point is that the difference arises in those cases in which the sequence for p contains only T's and V's. Thereby the sequence for $\sim p$ will, by the above rules, contain only F's and V's. Also, the sequence for $\Diamond\sim p$ will contain only F's and V's. In addition, the sequence for $\sim\Diamond\sim p$, that is the negation of $\Diamond\sim p$, will contain only T's and V's. But if the sequence for p contains only T's and V's, the sequence for $\Box p$ contains only F's and V's (and in fact, it is different from that for $\sim\Diamond\sim p$). Again, if a formula, like α, expresses a logical law (i.e. if not sequences for statements of the given form contain F's), then no sequences for statements of the corresponding form '$\sim\Diamond\sim\alpha$' will contain F's, but some for the form $\Box\alpha$ (namely those where the sequences for α contain V's as well as T's) will contain F's. Therefore, $\Box\alpha$ will not express a law.

Both with the infinite matrix M_Q and with the 6-valued approximation to it, there is a certain awkwardness about the verification of the rule of detachment. The difficulty arises at the following point: If we look at Table 3, we can see the implication '$(V_y - T_t) \supset (F_y - T_t)$' has the value '$V_y - T_t$'. Actually it is possible for both p and '$p \supset q$' to have the designated value '$V_y - T_t$' when q has the undesignated value '$F_y - T_t$'. Or using our more precise semantic valuing, p and '$p \supset q$' could both have the value 'VT' (the first symbol expresses the value of yesterday's and the latter is today's) when q has the value 'FT'. For example, 'I exist' and 'If I exist then someone other than God exists' could both be true today but could be vague yesterday (and therefore true whenever statable), even if 'Someone other than God exists' was yesterday definitely false. Similarly, with M_Q, the proposition p and the implication '$p \supset q$' might both have designated sequences (with no falsity), i.e. might both be true-whenever-statable, and q an undesignated sequence, provided that the q-sequence has a F only at places where the p-sequence has a V (i.e. provided that q is only false at the times at which p is vague).

It shall be stressed that this fact does not mean that it is ever unsafe to pass from a specific proposition (like p) and a specific implication (like $p \supset q$) to the corresponding proposition (that is, q). For example, the inference of 'Someone other than God exists' from 'I exist' and 'If I exist then someone other than God exists' is a perfectly valid one;

- at any time at which the premisses are true and the conclusion is true as well, and
- when the conclusion is *not* true, the premisses are vague, and the inference [not merely ought not to be but] just cannot be made.

It is, however, definitely untrustworthy to pass from the fact that a specific p (e.g., 'I exist') and a specific '$p \supset q$' (e.g., 'If I exist then someone other than God exists') are both true whenever statable, to the conclusion that q (e.g., 'Someone other than God exists') is true whenever statable.

III. A Formal-Logical Analysis of Q

What bearing has this upon the rule of detachment? It should be emphasised at the outset that the rule of detachment is 'not' the rule that from a specific p and '$p \supset q$' we may infer q nor is it the rule that from the truth-whenever-statable of a specific p and '$p \supset q$' we may infer the truth-whenever-statable of q. The following important proposition shall be taken into account.

Proposition. The rule of detachment is the rule that if (i) the formula α is so constructed that every proposition of this form has a designated value, and (ii) the formula '$\alpha \supset \beta$' (where α is as before) also has this property, then β has this property.

Let us focus on '$\Box p \supset \sim\Diamond\sim p$' now. The rule of detachment in Q would enable us to pass from the fact that anything of the form '$\Box p \supset \sim\Diamond\sim p$' is true whenever it is statable, and that anything of the form '$(\Box p \supset \sim\Diamond\sim p) \supset (\Diamond\sim p \supset \sim\Box p)$' is true whenever statable, to the conclusion that anything of the form '$\Diamond\sim p \supset \sim\Box p$' is true whenever statable. Note that '$(\Box p \supset \sim\Diamond\sim p) \supset (\Diamond\sim p \supset \sim\Box p)$' is obtainable by substitution in '$(p \supset \sim q) \supset (q \supset \sim p)$'. If our value-tables are such that no two propositions (in the forms of 'p' and '$p \supset q$') both take a designated value (unless q does), the rule of detachment will obviously hold, and with most systems this is how we show it will hold; but it also holds in some cases in which the value-tables are not of this sort, though the proof of it must then be more roundabout.

The position with regard to Q is as follows: Detachment can be shown to hold in Q itself, and it can also be shown to continue to hold when Q is enriched by name-variables, predicate variables, and quantifiers.[2] When, however, we enrich the symbolism

[2]Here Prior offers a proof for Q as follows: If there were a counter-example to detachment in Q it would be one in which every possible sequence for β which contained F's would contain them only

of Q with name-'constants' also, detachment can only be retained with the proviso that β must not be inferred from 'α' and '$\alpha \supset \beta$' if 'α contains a name-constant which does not occur in β'.[3] For example, suppose that a is the name of a certain non-sempiternal object and b is the name of a sempiternal object (which once existed alone). Consequently, the formula 'a is other than b' will have 'only one exemplification' (itself), and this will be true whenever statable. Correspondingly, the formula 'If a is other than b then something is other than b' will be in the same case; but the one exemplification of the formula 'Something is other than b' will be sometimes-false (and not always-true).

Among the rules and laws which are more obviously verified by our infinite matrix M_Q, it is important listing the following:[4]

RULES

- *R-I*: $\alpha \to \sim\Diamond\sim\alpha$
- *R-II*: $(\alpha \to \beta) \to (\Diamond\alpha \to \Diamond\beta)$, provided that there is no variable in β which is not in α
- *R-III*: $(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$, with the same proviso

LAWS

- *L-I*: $\Diamond p \supset \sim\Box\sim p$
- *L-II*: $\Box p \supset \sim\Diamond\sim p$
- *L-III*: $p \supset \Diamond p$
- *L-IV*: $(p \supset q) \supset (\Box p \supset \Box q)$
- *L-V*: $\Diamond\Box p \supset \Box p$

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where the corresponding sequence for α contained V's; the 'corresponding sequence for α' being that which results when all variables in α are assigned the same values as the same variables in β. Now a sequence for α can only contain a V where the sequence assigned to some variable in α has a V; and if this variable occurred also in β, with the same value assigned to it, it would cause the sequence for β to have a V in that place also. Hence this variable—call it v—cannot occur in β. But in that case there will be no way of ensuring, by the very structure of β, that β will have F's only where, and when, v has V's, for the variables on the values of which the value of β does depend will be ones different from v and so varying independently of it. (Prior finally states that this proof was hit upon independently by Dr. Alan Ross Anderson and him.)

Prior believes that when we introduce [and take into account] name-variables, there will be other elementary propositional formulae (i.e. ones not containing other propositional formulae) beside propositional variables. Accordingly, the capacity for independent variation of different elementary propositional formulae may be restricted by their containing the same name-variables (for at any point at which a sequence for $\Phi(x)$ contains a V, all simultaneously possible sequences for $\Psi(x)$ will also contain a V). However, Prior believes that the nature of the restriction makes it possible to extend the above proof to this extension of Q.

[3]The suggestion that this is the proviso required is due to Prior's wife.

[4]In this article the symbol '$\to$' is used to express how a law will be followed by another law. On the other hand, the symbol '$\supset$' represents the implication.

If we add to the afore-mentioned laws as a further law the formula '$\sim\Box\sim p \supset \Diamond p$', we can prove all of Gödel's postulates for S5. Therefore:

i. We have $\sim\Box\sim p \supset \Diamond p$. Accordingly, by substituting p for $\sim p$, we can conclude that $\sim\Box\sim\sim p \supset \Diamond\sim p$. Subsequently, by considering '$(\sim p \supset q) \supset (\sim q \supset p)$', we have $\sim\Diamond\sim p \supset \Box\sim\sim p$.

   Consequently—by '$\sim\sim p \supset p$', *R-III*, and syllogism—$\sim\Diamond\sim p \supset \Box p$ is concludable. Hence, by *R-I* and the last result, we have $\alpha \to \Box\alpha$.

ii. We have both implications '$\Diamond p \supset \sim\Box\sim p$' and '$\sim\Box\sim p \supset \Diamond p$'. Therefore, '$\Diamond$' and '$\sim\Box\sim$' are interchangeable in theses. The only doubt would be about modal contexts, but the proviso in *R-II* and *R-III* is automatically met when α and β only differ in one having $\Diamond$ where the other has $\sim\Box\sim$.

iii. We have $p \supset \Diamond p$. Therefore, by utilising '$(p \supset q) \supset (\sim q \supset \sim p)$', we can conclude that $\sim\Diamond p \supset \sim p$. Subsequently, by substituting p for $\sim p$, $\sim\Diamond\sim p \supset \sim\sim p$ is concludable. Now by taking into account the implication '$\sim\sim p \supset p$' we can have $\sim\Diamond\sim p \supset p$. Consequently, by *L-II*, we have $\Box p \supset p$.

iv. According to *L-IV*, we have $\Box(p \supset q) \supset (\Box p \supset \Box q)$.

v. Regarding '$\sim\Box\sim p \supset \Diamond p$', we have $(\Diamond\Box p \supset \Box p) \to (\sim\Box\sim p \supset \Box p)$. Subsequently, considering '$(\sim p \supset q) \supset (\sim q \supset p)$' we have $\sim\Box p \supset \Box\sim\Box p$.

Since any postulate-set sufficient for the tense-logical system Q must yield the rules *R-I*—*R-III* as well as the laws *L-I*—*L-V*, the addition of the law '$\sim\Box\sim p \supset \Diamond p$' to any such postulate-set will, by the result just gained, give a system equivalent to S5. Therefore, S5 may be thought of either:

- as the result of adding '$\Diamond\Box p \supset \Box p$' to a weaker Lewis system (say S2—see Ballarin (2017)); or
- as the result of adding '$\Diamond\Diamond p \supset p$' to the system formed by subjoining the rules M1, M2, L1, and L2 to Heyting's calculus (see Heyting (1930)); or
- as the result of adding '$\sim\Box\sim p \supset \Diamond p$' to a postulate-set sufficient for the tense-logical system Q.

That is in fact the system Q.

IV. Q from the Perspective of Tense Logic K_t

This section, based on the chapter 19 ("*Tense Logic for Non-Permanent Existents*") of Prior's "*Papers on Time and Tense*" (see Hasle *et al.* 2003), focuses on system Q from the perspective of tense logic. This section adds supportive logical concepts to the presented formal-logical analysis of Q. In addition, it offers a logical analysis of the relevant logical concepts from a different perspective. First, we focus on the howness of defining tense logic K_t by Prior. Accordingly, we review how Prior deals with Q in such a framework.

For Prior, tense logic is a kind of calculus in which the variables p, q, r, etc. stand for 'propositions' which may be true or false at different times. Correspondingly, in tense logic, the usual two-valued truth-functions (e.g., '$p \supset q$', '$p \vee q$', '$p \wedge q$', '$p \leftrightarrow q$', '$\sim p$') are supplemented by the two main forms Fp and Pp that stand for 'It will be the case that p' and 'It has been the case that p', respectively. In addition, the forms Gp (for 'It will always be the case that p') and Hp (for 'It has always been the case that p') are generally taken to be definable in terms of the others as $\sim F \sim p$ and $\sim P \sim p$, respectively. According to Hasle *et al.* (2003), it is sometimes argued that the 'propositions' of such a system are not properly so described, being in fact *predicates* of instants, its characteristic functors being used 'not to form propositions from propositions' but 'to form predicates from predicates'. We might use the variables a, b, c, etc. to stand for instants, and say that the theses of tense logic are all to be understood as predicable of any arbitrary instant a. We then can write pa for 'a is a p-ish instant' or 'The pseudo-proposition p is true at the instant a', and read $(Fp)a$, $(Pp)a$, $(p \supset q)a$, and $(\sim p)a$, similarly.

We are able to define our [tense-logic's] principal predicate-formers as follows:

1. $(p \supset q)a = (pa \Rightarrow qa)$
2. $(\sim p)a = \neg pa$
3. $(Fp)a = \exists b(Uab \ \& \ Pa)$
4. $(Pp)a = \exists b(Uba \ \& \ Pa)$,

where the symbols '$\Rightarrow$', '$\neg$' and '&' are used for truth functors of which the arguments are genuine propositions and $\exists b$ is utilised for an existential quantifier binding the instant-variable b. In addition, U is a two-place predicate which may be read informally as '...is earlier than...'.

Note that the definitions 3 and 4 assert that 'It is true at a that it will be the case that p' and 'It is true at a that it has been the case that p', respectively. In fact, they respectively mean that 'p is true at some instant later than a' and that 'p is true at some instant earlier than a'.

If we incorporate these definitions in an appropriate first-order theory, we may prove that certain tense-logical theses are predicable of ('true at') any arbitrary instant a. Prior states that E. J. Lemmon (see Lemmon 1956) has found that the tense-logical theses which have this property are precisely those which are derivable by substitution, detachment, and the rules:

- $a \to \sim F \sim a$,
- $a \to \sim P \sim a$.

Correspondingly, from some basis for propositional calculus and the axioms

1. $\sim F \sim (p \supset q) \supset (Fp \supset Fq)$

2. $\sim P \sim (p \supset q) \supset (Pp \supset Pq)$
3. $F \sim P \sim p \supset p$
4. $P \sim P \sim p \supset p$,

Prior defines the tense logic K_t.

Regarding Hasle *et al.* (2003), if certain conditions are imposed on the relation U, richer tense logics are obtainable (as predicable of an arbitrary instant a). For example, if we impose on U the 'transitivity', 'non-branching in the future', and 'non-branching in the past' conditions we may prove (as predicable of any arbitrary a) all the theses (and only the theses) of the tense logic obtained by adding to K_t the three further axioms

5. $FFp \supset Fp$
6. $PFp \supset (p \vee Fp \vee Pp)$
7. $FPp \supset (p \vee Fp \vee Pp)$.

In this strengthened system, if we define $\Diamond p$ as '$p \vee Fp \vee Pp$', we obtain for this $\Diamond$ the modal logic S5, which Lemmon (1956: 347) has shown to be the system obtainable by subjoining to propositional calculus, with substitution and detachment, the axiom '$p \supset \Diamond p$' and the RCM rule '$(\alpha \rightarrow \beta) \rightarrow (\Diamond\alpha \rightarrow \beta)$'.

Prior (1959) gives some postulates which he conjectured would suffice for a modal logic which takes account of this possibility and Prior (1964), by drawing upon a result of R. A. Bull, proves this conjecture correct. Actually Prior in his 'Tense Logic for Non-Permanent Existents' states that the modal logic in question is called Q, and its axiomatisation is as follows:

a. the ordinary modal form $\Diamond p$,
b. the form Sp, which may be taken as meaning 'In all possible worlds there is such a proposition as p', and
c. the form $\Box p$ as $Sp \wedge \sim\Diamond\sim p$.

For S we have the rules:

RS1. $S\alpha \rightarrow Sp$, for any p occurring in α

RS2. $Sp \rightarrow (Sq \rightarrow (... \rightarrow S\alpha))$, where $p, q, ...$ are all the variables in α.

For $\Diamond$ we have the axiom '$p \supset \Diamond p$' and the following modification of rule given in the last section for S5:

RSM. $(\alpha \rightarrow \beta) \rightarrow (Sp \rightarrow (... \rightarrow (\Diamond\alpha \rightarrow \beta))))$, where all the variables in β fall within the scope of a $\Diamond$ or an S, and $p, q, ...$ are all the variables in β that are not in α.

Subsequently, Prior, in such a logical framework, by introducing functors analogous to S, has argued that he shall propose modified postulates for the minimal tense logic K_t, and prove that if we strengthen this by adding suitable postulates for time's 'linearity' as well as 'the transitiveness of the earlier-later relation', and define $\Diamond p$ as $(p \vee Fp \vee Pp)$, and Sp in terms of the analogous functors, we obtain for this $\Diamond$ the modal system Q. For the full argumentation and logical analysis, please see chapter 19 of Hasle *et al.* (2003).

V. Concluding Remarks and Future Work

In his logical analysis of 'time and modality', Prior believed that there is adumbrated a modal system called Q, intended as a reasonably strong modal logic which would nevertheless lack such dubious principles as '$\sim\Box p \supset \Diamond\sim p$' and '$\sim\Diamond\sim(p \supset q) \supset (\Diamond p \supset \Diamond q)$', and which could be combined with a normal quantification theory without yielding the dubious principles in the mixed field. Q has not been axiomatised but, as analysed, been characterized by a matrix. Prior presented Q as a 'logic for contingent beings'; meaning by that a logic in which one could intelligibly say that some beings are contingent and some are necessary. Prior believed that there would still be very much more to be found out about system Q.

The future work will—by interpreting Q 'an ordinary modal logic' and/or 'a logic of necessity and possibility'—offer a longer discussion on System Q's powers and deficiencies. It will also focus on possible applications of Q in modern logical systems.

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