Anti-Diodorean Quantum Logic of Observables

Vladimir L. Vasyukov
Institute of Philosophy RAS
vasyukov4@gmail.com

Abstract. The conception of "Anti-Diodorean" logic is based on the reversal of the definitions of Diodorus Cronus, who proposed to define modal concepts in terms of temporal ones. In case of quantum logic this is done by using the semantic interpretation of anti-Diodorean definitions, i.e. semantically defining spatio-temporal concepts by means of a quantum accessibility relation. Since Quantum Logic of Observables could be treated as the many-valued extension of the usual Quantum Logic, we can consider the Anti-Diodorean Logic of Observables in the same way. DOI: 10.52119/LPHS.2024.33.38.010.
Keywords: Diodorean definition, logic of futurity, quantum logic of observables, causal logic.

Anti-Diodorean definition of the futurity operator F in terms of modalities $\Diamond$ and $\Box$ is as follows [1, p. 57]:
Definition 1. $F\alpha =_{def} \Box\Diamond\alpha$.
Taking into account that in modal systems, this definition semantically means that $x \models F\alpha \Leftrightarrow \forall y \exists z(xRy \,\&\, yRz \,\&\, z \models \alpha)$. In the semantics of orthologic, the accessibility relation xRy (x is not orthogonal to y) is reflexive and symmetric and the following theorems are proved [2, p. 140]:
Theorem. For any Kripkean realization for orthologic and any formula α:

$$x \models \alpha \Leftrightarrow \forall y \exists z(yRx \,\&\, zRy \,\&\, z \models \alpha) \tag{5}$$
$$x \models \neg\alpha \Leftrightarrow \forall y(yRx \,\&\, y \not\models \alpha) \tag{6}$$

Given the symmetry of accessibility relation in orthologic, it is easy to rewrite these theorems as

$$x \models \alpha \Leftrightarrow \forall y \exists z(xRy \,\&\, yRz \,\&\, z \models \alpha) \tag{7}$$
$$x \models \neg\alpha \Leftrightarrow \forall y(xRy \,\&\, y \not\models \alpha) \tag{8}$$

and comparing these formulations with the semantic condition of the *Definition* 6, it is not difficult to rewrite it as $x \models \alpha \Leftrightarrow x \models F\alpha$ and $x \models \neg\alpha \Leftrightarrow x \models \neg F\alpha$, i.e. according to the hypothesis on the temporal nature of orthologic. In [1] it is shown that this can be understood as the appearance of induced spacetime structure in all quantum processes.
If we try to extend our process of inducing spacetime structure on more sophisticated system of quantum logic, then in case of Quantum Logic of Observables [3] we need to analyze the semantic construction of another kind. It is quickly observed that an orthogonality relation below does not differ from that in case of quantum orthologic.
Definition 2. If X is a non-empty set, $\perp$ is an orthogonality relation on X if $\perp \subseteq X \times X$ is irreflexive and symmetric. We write $x \perp Y$ (where $x \in X, y \subseteq Y$) if for any $y \in Y$ we have $x \perp y$ and denote Y^* a set $\{x : x \perp y\}$. $Y \subseteq X$ is said to be $\perp$-closed if $Y^{**} = Y$. Besides this, $\emptyset \perp X$.
Definition 3. $\boldsymbol{H} = \langle X, \perp, \xi \rangle$ is a QLO-frame iff:
(1) X is a non-empty set (carrier of X);
(2) $\perp$ is an orthogonality relation on X;
(3) ξ is a non-empty collection of $\perp$-closed subsets of X closed under set intersection and the operation *.
Definition 4. $\boldsymbol{M} = \langle X, \perp, xi \rangle$ is a QLO-model if:

(i) $\langle X, \perp, \xi \rangle$ is a QLO-frame;
(ii) v is a function assigning to each propositional variable and formula of QLO recursively in every point (every element) of X a real number, i.e. $v : S \cup \Phi \times X \to \boldsymbol{R}$, where S is a set of propositional variables and Φ is a set of wff.
Denoting the set $\{xX : v(\alpha, x) = a\}$ by $||\alpha||_a$, we define recursively the value of a wff in QLO-model as follows:
(1) $||p_i||_a = \{x \in X : v(p_i,x) = a\} \in \xi$;
(2) $||\alpha \vee \beta||_a = \{x \in X : x \in ||\alpha||_b$ and $x \in ||\beta||_c$ and $a = b + c\}$;
(3) $||\alpha||_a = \{x \in X : x \in ||\alpha||_b$ and $x \in ||\beta||_c$ and $a = bc\}$;
(4) $||\neg\alpha||_a = \{x \in X : x \in ||\alpha||^*_{-a}$ and $v(\alpha, x) = a\}$;
(5) $||J_b\alpha||_c = \{x \in X : x \in ||\alpha||_a$ and $ba = c\}$;
(6) $||\mathbf{1}||_1 = X$, i.e. $v(\mathbf{1},x) = 1$ for all $x \in X$.
In the semantics of Quantum Logic of Observables we can define a Kripkean realization $x \models_a \alpha$ where $a \in \boldsymbol{R}$ is a value of observable in point and a value of a wff α in QLO-model as $||\alpha||_a = \{x \in X : x \models_a \alpha\}$. Then defining $y \models_{-a} \neg\alpha$ where y is orthogonal to x we obtain a value of a wff $\neg\alpha$ as $||\neg\alpha||_{-a} = \{y \in X : y \models_{-a} \neg\alpha$ and y is orthogonal to $x\}$. We also can prove the theorem

$$||\alpha||_a = ||F\alpha||_a$$
$$||\neg\alpha||_{-a} = ||\neg F\alpha||_{-a},$$

where $||F\alpha||_a = \{z \in X : \forall y \exists z(xRy \,\&\, yRz \,\&\, z \models_a a)\}$ and $||\neg F\alpha||_{-a} = \{y \in X : \forall y(xRy \,\&\, y \not\models \alpha)\}$. But immediately arising question concerns the nature of such "spacetimezation" of quantum observables.
First of all, it should be taken into account that if in case of quantum logic its algebraic structure is an ortho(modular) lattice while for the quantum logic of observables we have an algebra of positive elements of C^+-algebra to which belongs an algebra of observables. And for the ortho(modular) lattice the fact $x \models \alpha \Leftrightarrow x \models F\alpha$ could be understood as a transition to the semantics of causal space when the model of orthomodular logic becomes Minkowski spacetime, and the logic itself turns into the causal logic of Minkowski spacetime, where $x \models F\alpha$ should be read as "the causal path (trajectory) α will pass through x."
But even considering the transition from a Kripkean realization $x \models \alpha$ to a realization $x \models_a \alpha$ as in a sense preserving orthomodular structure, we cannot find the place for values of observables. It seems that we need to enrich the construction of the causal logic of Minkowski spacetime with the values of observables.
A model (M, G) of a causal space for quantum logic according to [1] is described as a pair where M is a non-empty set and G is a structure defined by a distinguished covering G of M by non-empty subsets. The elements $f \in G$ are called causal paths, and $S(x) = \{f \in G : x \in f\}$ is the set of all paths containing x. Two points x and y are causally related if there is some path f containing both of them.
For two points x and y from M, xRy means that x and y are causally related or simply connected while in quantum logic xRy means that x and y are not orthogonal. Orthocomplement in M is defined as $f^\perp = \{x : x$ is not connected with any point of $f\}$ and $f \in L(M)$ if and only if $f^{\perp\perp} = f$. For quantum logic the analogous definition looks like $f^\perp = \{x : x$ is orthogonal with any point of $f\}$. If we suggest that in any point of f we have one and the same $a \in \boldsymbol{R}$ then our f and $f^\perp$ will correspond to $||\alpha||_a$ and $||\neg\alpha||_{-a}$ respectively.

As a conjunction in $L(M)$ usually the intersection of sets $f \wedge g = f \cap g$ is used and from the point of view of quantum logic of observables it is easy come to the conclusion that this case gives us the treatment $||\alpha||_a \cap ||\beta||_b = \{x \in X : x \in ||\alpha||_a$ and $x \in ||\beta||_b$ and $ab = c\} = ||\alpha \wedge \beta||_c$. The specific of this definition as well as $||\alpha \vee \beta||_c = \{x \in X : x \in ||\alpha||_a$ and $x \in ||\beta||_b$ and $a + b = c\}$, $||J_b\alpha||_c = \{x \in X : x \in ||\alpha||_a$ and $ba = c\}$ is that they are local i.e. they do not exploit the relation R.

The situation looks like we're using a tensor field design but in spacetime tensors are four-dimensional constructions while we use only single numbers (one-dimensional constructions). Maybe it's worth to speak not of tensors but of scalar and scalar fields. So, in this case we in fact consider operations on scalar fields and it is worth to suggest that if we can introduce the definition of $f_a \wedge g_b$ not as the intersection not of causal paths but as product of scalar fields in spacetime.

References

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